Algebraic Noncommutative Geometry

Jonathan Gratus*
Physics Department, Lancaster University, Lancaster LA1 4YB

February 1, 2008

Abstract

A noncommutative algebra $\mathcal{A}$, called an algebraic noncommutative geometry, is defined, with a parameter ε in the centre. When ε is set to zero, the commutative algebra $\mathcal{A}^0$ of algebraic functions on an algebraic manifold $\mathcal{M}$ is obtained. This $\mathcal{A}^0$ is a subalgebra of $C^\omega(\mathcal{M})$, which is dense if $\mathcal{M}$ is compact. The generators of $\mathcal{A}$ define an immersion of $\mathcal{M}$ into $\mathbb{R}^n$, and $\mathcal{M}$ inherits a Poisson structure as the limit of the commutator. Thus $\mathcal{A}$ is a quantisation of a Poisson manifold. If an ordering convention is prescribed for $\mathcal{A}$ then a star product on $\mathcal{M}$ is obtained. Homomorphism and isomorphisms between noncommutative geometries are defined, and the map from $\mathcal{A}$ to the Heisenberg algebra is used both to give an analogue of a coordinate chart, and to give $\mathcal{A}$ a quantum group structure. Examples of algebraic noncommutative geometries are given, which include $\mathbb{R}^n$, $T^\star S^2$, T^2, S^2 and surfaces of rotation. A definition of a metric on $\mathcal{M}$ which can be extended to noncommutative geometry is given and this is used in an application of noncommutative geometry to the numerical analysis of surfaces.

Contents

*email: jg@luna.ph.lancs.ac.uk

Manifold	Section	Ordering	Star product	Representation	Trace (analytic)	Heisenberg Coordinates	Quantum Group
Euclidean flat space $\mathbb{R}^n$	3.1	Wick	Vey	∞-dimensional	no	yes	yes
		Normal	differential				
Phase space $T^\star S^2$	3.2			∞-dimensional	no	yes	no
Torus T^2 or Manin Plane	3.3	Normal	differential	Matrix	yes	yes	yes
		Central	Vey		(no)		
Surface of Rotation	3.4	Normal	non-diff	Matrix	yes	yes	yes
		Central	Vey		(yes)		
Sphere S^2	3.5	Normal	non-diff	Matrix	yes	yes	yes
		Central	Vey		(yes)		
		Wick-like	non-diff				
Plane $\mathbb{R}^2$ or $\mathbb{C}$	3.6			no	no	no	no

Table 1: The examples given in this article.

1 Introduction

Noncommutative geometry has been suggested as a method for the quantisation of gravity [6], string theory[5], renormalisation and a contribution to the elusive M-theory. There are two main goals in noncommutative geometry:

- Given a manifold or variety $\mathcal{M}$ the space of analytic functions $C^\omega(\mathcal{M})$ forms an infinite dimensional commutative algebra via pointwise multiplication. We wish to find a noncommutative algebra $\mathcal{A}$ which can be considered as the noncommutative analogue of $C^\omega(\mathcal{M})$.

  There are many possible principles to guide us to a definition of $\mathcal{A}$. Here, guided by quantum mechanics, we define an element ε in the centre of $\mathcal{A}$, which plays the rôle of $\hbar$. Thus when we set $\varepsilon = 0$ we obtain a new commutative algebra $\mathcal{A}^0$ which is a (dense) subalgebra of $C^\omega(\mathcal{M})$.
- We wish to define the tools of differential geometry, such as tangent spaces, differential forms, connections and curvature, in terms of the elements of $C^\omega(\mathcal{M})$, and then find analogues of these objects when $C^\omega(\mathcal{M})$ is replaced by $\mathcal{A}$, so that they regain there original definition when $\varepsilon = 0$.

Most of this paper is concerned with the first of these goals, and, having defined $\mathcal{A}$, giving detailed examples. The final section gives an application for this theory to the numerical analysis of surfaces.

The intrinsic method of defining a manifold is in terms of coordinate charts. However the method employed here is to assume that $\mathcal{M}$ is immersed in the real Euclidean space $\mathbb{R}^n$. This

implies that if $\dim(\mathcal{M}) = D$ then there are $n - D$ functions $\{\mathfrak{b}_1^0, \ldots, \mathfrak{b}_{n-D}^0\}$ with $\mathfrak{b}_s^0 \colon \mathbb{R}^n \mapsto \mathbb{R}$ such that

$$\mathcal{M} = \{\underline{x} \in \mathbb{R}^n \,|\, \mathfrak{b}_s^0(\underline{x}) = 0,\, \forall \mathfrak{b}_s^0\} \tag{1}$$

If the coordinates of $\mathbb{R}^n$ are given by $(x_1, \ldots, x_n)$ then each $x_i \in C^\omega(\mathcal{M})$. These are, in a certain sense, privilege elements of $C^\omega(\mathcal{M})$, as they encode all the information about $\mathcal{M}$. They are called immersion coordinates.

In this article we shall further assume that $\mathcal{M}$ is algebraic; that is, each $\mathfrak{b}_s^0(\underline{x})$ is a polynomial (multinomial) in the coordinates $(x_1, \ldots, x_n)$. Likewise we only consider the subalgebra $\mathcal{A}^0 \subset C^\omega(\mathcal{M})$ of polynomials in $(x_1, \ldots, x_n)$. By restricting ourselves to algebraic manifolds, we avoid many problems associated with convergence. However, we shall see by the list of examples that this still enables us to study a large class of interesting manifolds.

Since $\mathcal{A}^0$ is a commutative algebra, we have the commutation equations:

$$[x_i, x_j] = 0 \tag{2}$$

where the square bracket represents the commutator. Together with the immersion equations $\{\mathfrak{b}_s^0(\underline{x}) = 0\}$, this gives a total of $n(n-1)/2 + n - D$ equations, which completely specify $\mathcal{A}^0$.

The noncommutative algebra $\mathcal{A}$ is also specified in this way. It is generated by the immersion coordinates $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ and a parameter ε in the centre of $\mathcal{A}$. We use the bold font to specify that $\boldsymbol{x}_i \in \mathcal{A}$ and hence do not commute. We replace (2) with

$$[\boldsymbol{x}_i, \boldsymbol{x}_j] = i\varepsilon \boldsymbol{c}'_{ij} \tag{3}$$

for some $\boldsymbol{c}'_{ij} \in \mathcal{A}$. Thus when we set $\varepsilon = 0$ this reduces to (2). To make all this mathematically precise we define everything by quotienting the algebra $\mathcal{F}$, which is the free noncommuting algebra generated by $(\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \varepsilon)$, by various ideals. The Euler font is used for elements of $\mathcal{F}$. The details of this are in section 2.1.

We call the algebra $\mathcal{A}$ an algebraic noncommutative geometry, abbreviated to ANCG, to distinguish it from the noncommutative geometry of Connes [3, 4] and the matrix geometry of Madore [13].

The act of setting $\varepsilon = 0$ is given by the quotient map $\pi \colon \mathcal{A} \mapsto \mathcal{A}^0$. This may be considered as *classicisation* i.e. taking one from a quantum system to a classical system. The first consequence of this definition is that $\mathcal{M}$ inherits a Poisson structure. This is given, in section 2.2, by

$$\{\pi(\boldsymbol{u}), \pi(\boldsymbol{v}\}) = \pi(\tfrac{1}{i\varepsilon}[\boldsymbol{u}, \boldsymbol{v}])$$

where $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}$. Thus we have a method for the quantisation of immersed Poisson manifolds.

In section 2.3 we consider orderings. Since $\boldsymbol{u}\boldsymbol{v} \neq \boldsymbol{v}\boldsymbol{u}$, then, in order to specify the quantum analogue of a particular classical function, we must specify an ordering convention. This is given by a linear map $\Omega : \mathcal{A}^0 \mapsto \mathcal{A}$, so that $\pi \circ \Omega = 1_{\mathcal{A}^0}$. In quantum mechanics one often only has to specify an ordering for the Hamiltonian. One of the advantages with our approach to quantisation, is that we do not, a priori, assume an ordering convention, and can therefore compare different ordering conventions on the same algebra $\mathcal{A}$. For example for the Heisenberg algebra $\mathcal{H}_2$ where $[\boldsymbol{p}, \boldsymbol{q}] = i\varepsilon$, which underlies the quantum mechanics of a free particle on a line, we often consider two orderings:

Wick ordering;	$\Omega_W(p^r q^s) =$ sum of symmetric permutations of $\boldsymbol{p}^r \boldsymbol{q}^s$
Normal ordering;	$\Omega_N(p^r q^s) = \boldsymbol{p}^r \boldsymbol{q}^s$

where $\pi(\boldsymbol{p}) = p$ and $\pi(\boldsymbol{q}) = q$. Once we fix a particular ordering then the algebra $(\mathcal{A}, \Omega)$ is equivalent to a star algebra [1]; that is, we can transform the product on $\mathcal{A}$ to a star product given by

$$u \star v = \sum_{r=0}^{\infty} \varepsilon^r C_r(u, v) \tag{4}$$

where $C_r : \mathcal{A}^0 \times \mathcal{A}^0 \mapsto \mathcal{A}^0$. If one chooses the Wick ordering of $\mathcal{H}_2$ then we have a Vey product where $C_r = (\frac{i}{2}\mathcal{P})^r/r!$ where $\mathcal{P}(u, v) = \{u, v\}$. On the other hand, if one chooses the normal ordering of $\mathcal{H}_2$ then one has a different differential star product.

Of course star products can be given intrinsically, simply by specifying the functions $C_r\colon C^\omega(\mathcal{M}) \times C^\omega(\mathcal{M}) \mapsto C^\omega(\mathcal{M})$. We have the following pseudo equation:

$$\text{Algebraic Noncommutative Geometry} + \text{Ordering} \approx \text{Star Product} + \text{Immersion Coordinates} \tag{5}$$

This is not strictly true though, due to our restriction to algebraic functions.

Completely separate to the question of which ordering to impose, is the question of whether representations of $\mathcal{A}$ exist. This is investigated in section 2.5. If $\mathcal{M}$ is compact then there may exists a sequence of matrix representation of $\mathcal{A}$. These are maps $\varphi_N\colon \mathcal{A} \mapsto M_n(\mathbb{C})$, such that $\varphi_N(\varepsilon) = \varepsilon_N \mathbf{1}_N$, with $\varepsilon_N \in \mathbb{R}$, $\varepsilon_N \to 0$ as $N \to \infty$. The algebra of matrices which are the image of φ_N can be thought of as a matrix geometry. For compact symplectic manifolds the limit of the trace can be written in terms of the integral over $\mathcal{M}$.

$$\lim_{N\to\infty} \tfrac{1}{N}\mathrm{tr}(\varphi_N(f)) = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} \pi(f)\omega^n$$

where ω is the symplectic 2-form and $|\mathcal{M}| = \int_{\mathcal{M}} \omega^n$.

In section 2.4 we define the concept of homomorphisms and isomorphisms between noncommutative geometries. An important case is when the codomain is the Heisenberg algebra, which is the noncommutative analogue of Euclidean flat space. This mapping may then be considered as the noncommutative analogue of a coordinate system (section 2.6). We give examples of such noncommutative coordinate systems. This returns us to the original ideas of Dirac who suggested that one could consider manifolds where the coordinates do not commute. Since the Heisenberg algebra can be given the structure of a quantum group we can use the coordinate homomorphisms to give the quantum group structure to other noncommutative geometries (section 2.7).

In section 3, we give a number of examples. These are the cotangent bundle of the sphere, flat space and the two dimensional manifolds of the plane, torus, sphere, and surfaces of rotation. A list of properties is given is table 1. The cotangent bundle, section 3.2, should be thought of as the noncommutative analogue of a phase space. Thus we demonstrate that algebraic noncommutative geometry is indeed a method of quantisation. This gives the underlying quantum algebra corresponding to the non-relativistic quantisation of a free particle on a sphere.

As mentioned the second goal of noncommutative geometry is to write down the objects studied in differential geometry, such as tangent bundles, cotangent bundles, exterior algebras, metric tensors, connections and curvature, in terms of elements of the algebra $C^\omega(\mathcal{M})$ and then

find analogues of these objects when $C^\omega(\mathcal{M})$ is replaced by $\mathcal{A}$, so that they regain there original definition when $\varepsilon = 0$.

There are two key properties required of tangent vector fields. Firstly that they should be derivatives, i.e. follow Leibniz rule, and secondly that they should form a module over the algebra of functions. (That is one can multiply a vector with a scalar to give another vector.) It turns out that for noncommutative geometry these two properties are incompatible, and one must choose either to have vectors which are derivatives, or vectors which form a module.

The standard method is to choose vectors which form derivatives [13]; that is, $\xi : \mathcal{A} \mapsto \mathcal{A}$ such that $\xi(\boldsymbol{u}\boldsymbol{v}) = \xi(\boldsymbol{u})\boldsymbol{v} + \boldsymbol{u}\xi(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}$. For many noncommutative geometries we can show that this implies that all vector are inner, i.e. there exists $\boldsymbol{w} \in \mathcal{A}$ such that $\xi = \frac{1}{\varepsilon}\mathrm{ad}_{\boldsymbol{w}}$ where $\mathrm{ad}_{\boldsymbol{w}}(\boldsymbol{u}) = [\boldsymbol{w}, \boldsymbol{u}]$. Clearly if ξ is inner then $\boldsymbol{u}\xi$ is not inner.

In [8, 10] the author gives an alternative method of defining tangent vectors on the noncommutative sphere and surfaces of rotation. These vectors do form a (one sided) module over the noncommutative surface but are derivatives only in the commutative limit; that is, $\xi(\boldsymbol{u}\boldsymbol{v}) = \xi(\boldsymbol{u})\boldsymbol{v} + \boldsymbol{u}\xi(\boldsymbol{v}) + O(\varepsilon)$.

In this article, section 2.9, we circumvent the problem of how to define a vector field by defining the objects in differential geometry using only the elements of $C^\omega(\mathcal{M})$. This we do by writing the metric on $\mathcal{M}$ as $g(du^\sharp, dv^\sharp)$, where g is the metric inherited from the ambient Euclidean immersion space and $\sharp : T^\star\mathcal{M} \mapsto T\mathcal{M}$ is the metric dual. We show that this expression can be written using only the Poisson structure, the immersion coordinates, and the local coordinates on $\mathcal{M}$.

In section 4, we outline a method for the numerical analysis of surfaces embedded in $\mathbb{R}^3$. Let us assume we wish to analysis a surface, $\mathcal{M}$, which is nearly spherical; that is, the function $\{x_1, x_2, x_3\}$, when expanded in spherical harmonics, converges quickly. It therefore makes sense to use this property in any numerical analysis of $\mathcal{M}$, and to encode the information about the problem in terms of spherical harmonics as opposed to pointwise encoding.

Let us assume we wish to calculate simply $u = vw$ where $u, v, w \in C^\omega(\mathcal{M})$. We can express these functions using spherical harmonics as $u = \sum_{nm} u_{nm}\psi^n_m$ etc. From the Echart-Wigner theorem we have

$$u_{nm} = \sum_{m_1,n_1,n_2} v_{n_1 m_1} w_{n_2,m_2} C^{n_1\, n_2\, \quad n}_{m_1\, m-m_1\, m} C^{n_1 n_2 n}_{0\ 0\ 0}$$

Now if we work numerically then we truncate this sum and loose all modes ψ^n_m for $n \geq N$, for some $N \in \mathbb{N}$. As a result, in general $(uv)w \neq u(vw)$. Thus the corresponding algebra, although it is commutative, is non-associative.

By contrast, we propose, in section 4, to use the noncommutative spherical harmonics $\boldsymbol{P}^m_n$ described in section 3.5. The corresponding algebra is associative but noncommutative, indeed for numerical work this algebra is simply the algebra of $N \times N$ matrices. The method uses the results described in this article so that differentiation is replaced by commutation, and integration is replaced by trace.

All the information lost (or error) is introduced when we convert functions u on $\mathcal{M}$ into $N \times N$ matrices. After this, we can multiply any number of matrices without loosing additional information. Although the answer depends on the ordering of the expression we wish to calculate, we will show that any difference will be of order $O(1/N)$.

Finally in section 5 we discuss some of the possible methods of enlarging $\mathcal{A}$ so that $\mathcal{A}^0$ includes all analytic function on $\mathcal{M}$. We also discuss other developments of this theory and possible applications in physics.

NOTE: This article is arranged so that all the theorems are stated and proved before the main examples are given. This may not be the easiest way to read this article and the casual reader is recommended to scan the examples is section 3 before and whilst reading the theorems in section 2.

1.1 Notation

In this article we have a number of algebras, with many maps between them. The elements in the main algebras are written with different scripts to aid understanding. These are given by

algebra	generators	general element
$\mathcal{F}$	$\{\varepsilon, \mathfrak{x}_1, \dots, \mathfrak{x}_n\}$	$\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \mathfrak{y}$
$\mathcal{A}$	$\{\varepsilon, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n\}$	$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{y}$
$\mathcal{A}^0$	$\{x_1, \dots, x_n\}$	u, v, w, y
$\mathcal{A}^\star$	$\{\varepsilon, \dot{x}_1, \dots, \dot{x}_n\}$	$\dot{u}, \dot{v}, \dot{w}, \dot{y}$

The expression $C^\omega(\mathcal{M})$ refers to the algebra of complex valued analytic functions on $\mathcal{M}$. This means that for each function in $C^\omega(\mathcal{M})$ there is a Taylor expansion about each point in $\mathcal{M}$.

The term **polynomial** in $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ means any expression generated by taking finite sums and products.

The square brackets always refer to the commutator, so that $[\boldsymbol{u}, \boldsymbol{v}] = \boldsymbol{uv} - \boldsymbol{vu}$.

When talking about elements of $\mathcal{A}$ we will use the notation $\boldsymbol{u} = O(\varepsilon^r)$ for $r \in \mathbb{N}$ to mean $\boldsymbol{u} = \varepsilon^r \boldsymbol{u}'$ for some $\boldsymbol{u}' \in \mathcal{A}$. Likewise for elements of $\mathcal{F}$ and $\mathcal{A}^\star$.

Algebraic noncommutative geometry is abbreviated to ANCG, whilst ordered algebraic noncommutative geometry is abbreviated to OANCG.

2 Definition and Properties of Algebraic Noncommutative Geometries

2.1 Definition of ANCG

Let $\mathcal{F}^0$ be the free associative noncommutative complex algebra with a unit, finitely generated by $\{\mathfrak{x}_1, \dots, \mathfrak{x}_n\}$. Let $\mathcal{F}^{(\text{fin})}$ and $\mathcal{F}^{(\infty)}$ be the algebra with elements

$$\mathcal{F}^{(\text{fin})} = \left\{ \sum_{r=0}^{\text{finite}} \varepsilon^r \mathfrak{u}_r \,\middle|\, \mathfrak{u}_r \in \mathcal{F}^0 \right\} \qquad \mathcal{F}^{(\infty)} = \left\{ \sum_{r=0}^{\infty} \varepsilon^r \mathfrak{u}_r \,\middle|\, \mathfrak{u}_r \in \mathcal{F}^0 \right\} \tag{6}$$

where ε is in the centre of $\mathcal{F}^{(\text{fin})}$ and $\mathcal{F}^{(\infty)}$. The notation $\displaystyle\sum_{r=0}^{\text{finite}}$ means a finite sum over non negative r. Unless otherwise specified, we write $\mathcal{F}$ to mean either $\mathcal{F}^{(\text{fin})}$ or $\mathcal{F}^{(\infty)}$.

Since ε is in the centre of $\mathcal{F}$ we can define the map $\pi^F \colon \mathcal{F} \mapsto \mathcal{F}^0 = \mathcal{F} / \{\varepsilon \sim 0\}$. Thus π^F is the equivalent to setting $\varepsilon = 0$. If $\mathfrak{u}$ is written as (6) then $\pi^F(\mathfrak{u}) = \mathfrak{u}_0$.

We also define the quotient algebra and quotient map

$$Q^{C0} \colon \mathcal{F}^0 \mapsto \mathcal{F}^{C0} = \mathcal{F}^0 / \left\{ [\mathfrak{x}_i, \mathfrak{x}_j] \sim 0 \right\} \tag{7}$$

So $\mathcal{F}^{C0}$ is the free commutative algebra generated by $\{x_1, \dots, x_n\}$ where $x_i = Q^{C0}(\pi^F(\mathfrak{x}_i))$. We specify that $x_i = \overline{x_i}$ so that $\mathcal{F}^{C0}$ is a subalgebra of $C^\omega(\mathbb{R}^n)$.

We define an **Algebraic Noncommutative geometry** $\mathcal{A}$ as a quotient algebra of $\mathcal{F}$. When we need to be precise we will write $\mathcal{A}^{(\mathrm{fin})}$ or $\mathcal{A}^{(\infty)}$ if it is the quotient of $\mathcal{F}^{(\mathrm{fin})}$ or $\mathcal{F}^{(\infty)}$ respectively.

- $\mathcal{A}$ is noncommutative and associative.
- $\mathcal{A}$ is the quotient of the algebra $\mathcal{F}$, for some $n \in \mathbb{N}$, via the ideal generated from quotient elements

$$\mathfrak{c}_{ij} \in \mathcal{F}\,, \qquad i,j = 1,\dots,n \tag{8}$$

$$\mathfrak{b}_s \in \mathcal{F}\,, \qquad s = 1,\dots,n-D \tag{9}$$

  for some $D \in \mathbb{N}$, $D \leq n$. The ideal is all elements of the form

$$\sum_{i,j=1}^{n} \mathfrak{u}_{ij}\mathfrak{c}_{ij} + \sum_{s=1}^{n-D} \mathfrak{v}_s \mathfrak{b}_s\,, \qquad \mathfrak{u}_{ij}, \mathfrak{v}_s \in \mathcal{F}$$

  Since $\mathcal{A}$ is associative, this ideal must be a two sided ideal. The quotient map is written

$$Q \colon \mathcal{F} \mapsto \mathcal{A} \ \text{ with } Q(\varepsilon) = \varepsilon\,,\ Q(\mathfrak{x}_i) = \boldsymbol{x}_i \tag{10}$$

- Since ε is in the centre of $\mathcal{A}$ we can quotient $\mathcal{A}$ by the ideal generated from ε. This is equivalent to setting $\varepsilon = 0$. Thus we define the map

$$\pi \colon \mathcal{A} \mapsto \mathcal{A}^0 = \mathcal{A}/\left\{\varepsilon \sim 0\right\} \qquad \text{with } \pi(\varepsilon) = 0\,,\ \pi(\boldsymbol{x}_i) = x_i \tag{11}$$

- The commutation quotient relations $\mathfrak{c}_{ij}$ obey

$$\mathfrak{c}_{ij} = \mathfrak{x}_i\mathfrak{x}_j - \mathfrak{x}_j\mathfrak{x}_i - i\varepsilon\mathfrak{c}'_{ij} \qquad \text{where } \mathfrak{c}'_{ij} \in \mathcal{F} \tag{12}$$

  and where there is at least one $\mathfrak{c}'_{ij}$ such that $\pi(Q(\mathfrak{c}'_{ij})) \neq 0$.
- The immersion quotient relations $\mathfrak{b}_s$ obey

$$Q^{C0}(\pi^{F}(\mathfrak{b}_s)) \neq 0 \qquad \forall s = 1,\dots,n-D \tag{13}$$

- $\mathcal{A}$ be a conjugation algebra; that is, there exists a conjugation $\dagger \colon \mathcal{A} \mapsto \mathcal{A}$, where

$$(\boldsymbol{u}\boldsymbol{v})^\dagger = \boldsymbol{v}^\dagger\boldsymbol{u}^\dagger\,,\quad \boldsymbol{u}^{\dagger\dagger} = \boldsymbol{u}\,,\quad \varepsilon^\dagger = \varepsilon\,,\quad (\boldsymbol{x}_i)^\dagger = \boldsymbol{x}_i\,,\quad \lambda^\dagger = \overline{\lambda}\,, \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \in \mathcal{A}\,,\ \lambda \in \mathbb{C} \tag{14}$$

There are several more algebras, all of which are quotients of $\mathcal{F}$ which are useful. These are defined as follows, with their corresponding quotient maps.

$$\begin{array}{ll}
Q^{C} \colon \mathcal{F} \mapsto \mathcal{A}^{C} = \mathcal{F}/\left\{\mathfrak{c}_{ij} \sim 0\right\} & \\
Q^{I} \colon \mathcal{A}^{C} \mapsto \mathcal{A} = \mathcal{A}/\left\{\mathfrak{b}_s \sim 0\right\} & \text{so that } Q = Q^{I} \circ Q^{C} \\
Q^{C0} \colon \mathcal{F}^0 \mapsto \mathcal{F}^{C0} = \mathcal{F}/\left\{[\mathfrak{x}_i,\mathfrak{x}_j] \sim 0\right\} & \\
Q^{I0} \colon \mathcal{F}^0 \mapsto \mathcal{A}^0 = \mathcal{F}^0/\left\{\mathfrak{b}^0_s \sim 0\right\} & \text{where } \mathfrak{b}^0_s = \pi^{F}(Q^{C0}(\mathfrak{b}_s)) \\
\pi^{F} \colon \mathcal{F} \mapsto \mathcal{F}^0 = \mathcal{F}/\left\{\varepsilon \sim 0\right\} & \\
\pi^{C} \colon \mathcal{A}^{C} \mapsto \mathcal{F}^{C0} = \mathcal{A}^{C}/\left\{\varepsilon \sim 0\right\} &
\end{array} \tag{15}$$

The algebras $\mathcal{F}, \mathcal{F}^0, \mathcal{F}^{C0}$ depend only on n, whilst $\mathcal{A}^{C}, \mathcal{A}, \mathcal{A}^0$ depend on n and the quotient relations $\mathfrak{c}_{ij}$ and $\mathfrak{b}_s$. Since all the maps simply correspond to quotients they are related via the following commutative diagram.

$$\begin{array}{ccccc}
\mathcal{F} & \xrightarrow{Q^{C}} & \mathcal{A}^{C} & \xrightarrow{Q^{I}} & \mathcal{A} \\
\downarrow^{\pi^{F}} & & \downarrow^{\pi^{C}} & & \downarrow^{\pi} \\
\mathcal{F}^0 & \xrightarrow{Q^{C0}} & \mathcal{F}^{C0} & \xrightarrow{Q^{I0}} & \mathcal{A}^0
\end{array} \tag{16}$$

Since $\mathcal{F}^{C0} \subset C^\omega(\mathbb{R}^n)$, we can write $\mathfrak{b}_s^0 \colon \mathbb{R}^n \mapsto \mathbb{C}$. Let

$$\mathcal{M} = \left\{ \underline{x} \in \mathbb{R}^n \,\middle|\, \mathfrak{b}_s^0(\underline{x}) = 0, s = 1, \dots, n - D \right\} \tag{17}$$

In general $\mathcal{M}$ is an algebraic verity. If there are no critical points then $\mathcal{M}$ is a manifold. Thus $\mathcal{A}^0 \subset C^\omega(\mathcal{M})$ is the commutative subalgebra of complex algebraic function on $\mathcal{M}$.

Since $x_i = \overline{x_i}$ for $x_i \in \mathcal{F}^{C0}$ then $x_i = \overline{x_i}$ for $x_i \in \mathcal{A}^0$. It is easy to show that this implies the π preserves conjugation; that is, $\pi(\boldsymbol{u}^\dagger) = \overline{\pi(\boldsymbol{u})}$ for $\boldsymbol{u} \in \mathcal{A}$.

Lemma 1. *If $\mathcal{M}$ is compact then $\mathcal{A}^0$ is dense in the space $C(\mathcal{M})$ of continuous complex valued functions on $\mathcal{M}$ with the uniform norm*

$$\|f\| = \sup_{x \in \mathcal{M}} |f(x)| \tag{18}$$

Proof. Follows from the Boltzano-Wiestrass theorem. □

2.2 Poisson Structure

Theorem 2. *There exists a Poisson structure on $\mathcal{A}^0$ given by*

$$\{\bullet, \bullet\} : \mathcal{A}^0 \times \mathcal{A}^0 \mapsto \mathcal{A}^0 ; \qquad \{\pi(\boldsymbol{u}), \pi(\boldsymbol{v})\} = \pi(\tfrac{1}{i\varepsilon}[\boldsymbol{u}, \boldsymbol{v}]) \tag{19}$$

This can be extended to a Poisson structure on $\mathcal{M}$ given by $\{\bullet, \bullet\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \mapsto C^\infty(\mathcal{M})$.

Proof. Given $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}$, then since π is a homomorphisms, $\pi([\boldsymbol{u}, \boldsymbol{v}]) = [\pi(\boldsymbol{u}), \pi(\boldsymbol{v})] = 0$. Thus $[\boldsymbol{u}, \boldsymbol{v}] = O(\varepsilon)$, so $\frac{1}{\varepsilon}[\boldsymbol{u}, \boldsymbol{v}] \in \mathcal{A}$. Hence the Poisson bracket is defined. Given $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{A}$ such that $\pi(\boldsymbol{u}_1) = \pi(\boldsymbol{u}_2)$ then $\boldsymbol{u}_1 - \boldsymbol{u}_2 = O(\varepsilon)$. Hence $[\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{v}] = O(\varepsilon^2)$, so $\{u_1, v\} = \{u_2, v\}$, and the Poisson bracket is well defined. From (12) we know there exist $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}$ such that $\pi(\frac{1}{i\varepsilon}[\boldsymbol{u}, \boldsymbol{v}]) \neq 0$, and hence the Poisson bracket is non trivial.

The derivative property follows from expanding $\pi(\frac{1}{i\varepsilon}[\boldsymbol{u}, \boldsymbol{v}\boldsymbol{w}])$. The Jacobi identity follows from the Jacobi identity for commutators.

Since all Poisson structures my be written in terms of a bivector, the Poisson structure can be extended to $C^\infty(\mathcal{M})$. □

We say that $\mathcal{A}$ is **symplectic**, if the Poisson structure on $\mathcal{M}$ is symplectic.

2.3 Orderings and Star Products

An **ordering** on a ANCG $(\mathcal{A}, \mathcal{M})$ is a choice of injective linear map $\Omega \colon \mathcal{A}^0 \mapsto \mathcal{A}$ such that $\pi \circ \Omega = 1_{\mathcal{A}^0}$ and $\Omega(1) = 1$. Furthermore, Ω is a **unitary ordering** if $\Omega(\overline{u}) = \Omega(u)^\dagger$.

The following theorem proves that all ANCG possess at least one ordering. In general for a given ANCG there will be an infinite number of orderings Ω.

Theorem 3. *Given a ANCG $(\mathcal{A}, \mathcal{M})$ there exists a non unique unitary ordering Ω.*

Proof. Choose a sequence of self-conjugate elements in $\mathcal{F}$ which is a basis of $\mathcal{F}^0$ as an ∞-dimensional vector space. For example

$$\mathfrak{x}_1, \mathfrak{x}_2, \dots, \mathfrak{x}_n, \mathfrak{x}_1\mathfrak{x}_1, \mathfrak{x}_1\mathfrak{x}_2, \dots, \mathfrak{x}_1\mathfrak{x}_n, \mathfrak{x}_2\mathfrak{x}_1, \dots, \mathfrak{x}_2\mathfrak{x}_n, \dots, \mathfrak{x}_n\mathfrak{x}_1, \dots \mathfrak{x}_n\mathfrak{x}_n, \mathfrak{x}_1\mathfrak{x}_1\mathfrak{x}_1 \dots, \dots \tag{20}$$

now remove any elements from the sequence that are permutation of previous elements, or are in the span of preceding elements and the ideal generated by the quotients elements $\mathfrak{b}_s^0$.

This gives a sequence of $\mathfrak{u}_i \in \mathcal{F}$. Each element in $\mathcal{A}^0$ can be uniquely written as a sum $v = \sum_{i=0}^{\text{finite}} \lambda_i \pi(Q(\mathfrak{u}_i))$. Set $\Omega(\boldsymbol{v}) = \sum_{i=0}^{\text{finite}} \lambda_i Q(\mathfrak{u}_i)$.

To construct a unitary ordering set $\Omega_U(v) = \frac{1}{2}(\Omega(v) + \Omega(\overline{v})^\dagger)$.

Ω is far from unique. For example we can always set a new ordering as $\Omega_1(v) = \Omega(v) + \varepsilon$. □

There are certain orderings that have been given names. For example the **Wick ordering** and **normal ordering**. However the exact definition of these orderings depend on the algebra $\mathcal{A}$.

We define an **Ordered Algebraic Noncommutative Geometry** (OANCG) as an ANCG $\mathcal{A}$ together with a choice of orderings Ω. We can now give the relation between OANCG and star products. We define the set $\mathcal{A}^\star$ as the set of all elements of the form

$$\mathcal{A}^\star = \left\{ \dot{u} = \sum_{r=0}^{\infty} \varepsilon^r u_r \middle| \quad u_r \in \mathcal{A}^0 \right\} \tag{21}$$

Theorem 4. *We can extend the ordering Ω to give the map*

$$\tilde{\Omega} \colon \mathcal{A}^\star \mapsto \mathcal{A}\,; \qquad \tilde{\Omega}\left(\sum_{r=0}^{\infty} \varepsilon^r u_r\right) = \sum_{r=0}^{\infty} \varepsilon^r \Omega(u_r) \tag{22}$$

This map has an inverse given by

$$\tilde{\Omega}^{-1} \colon \mathcal{A} \mapsto \mathcal{A}^\star\,; \qquad \tilde{\Omega}^{-1} \colon \boldsymbol{u} \mapsto \sum_{r=0}^{\infty} \varepsilon^r C_r(\boldsymbol{u}) \tag{23}$$

where $C_r \colon \mathcal{A} \mapsto \mathcal{A}^0$ is given by

$$C_r(\boldsymbol{u}) = \pi\left(\varepsilon^{-n}\left(\boldsymbol{u} - \sum_{m=0}^{r-1} \varepsilon^m \Omega(C_m(\boldsymbol{u}))\right)\right) \tag{24}$$

This satisfies $\tilde{\Omega} \circ \tilde{\Omega}^{-1} = 1_{\mathcal{A}}$ and $\tilde{\Omega}^{-1} \circ \tilde{\Omega} = 1_{\mathcal{A}^\star}$

Proof. Trivial. □

Given an OANCG we can define a **star product** on $\mathcal{A}^0$. This is given by

$$\star \colon \mathcal{A}^0 \times \mathcal{A}^0 \mapsto \mathcal{A}^\star\,; \qquad u \star v = \tilde{\Omega}^{-1}\Big(\Omega(u)\Omega(v)\Big) = \sum_{r=0}^{\infty} \varepsilon^r C_r(u,v) \tag{25}$$

where

$$C_r(u,v) = C_r(\Omega(u)\Omega(v)) \tag{26}$$

We note that $C_0(u,v) = uv$ and $C_1(u,v) - C_1(v,u) = i\,\{u,v\}$. We extend (25) to the star product $\star : \mathcal{A}^\star \times \mathcal{A}^\star \mapsto \mathcal{A}^\star$. We call the set $\mathcal{A}^\star$ together with the product $\star$, a **star algebra**. This makes $\tilde{\Omega} : \mathcal{A} \mapsto \mathcal{A}^\star$ a bijective homomorphism. A **differentiable star product** requires that $C_r(u,v)$ is a bi-differential of u and v. A **Vey Product** is a differentiable star product where $C_r = (\frac{i}{2}\mathcal{P})^r/r!$ where $\mathcal{P}$ is the bi-differential operator given by $\mathcal{P}(u,v) = \{v,u\}$.

Counter example 1: We note that, if we have a ε-finite OANCG, given by $\mathcal{A} = \mathcal{A}^{(\text{fin})}$, then in general, we still require an infinite expansions in ε in $\mathcal{A}^\star$. To see this consider the noncommutative complex disk [12], generated by $\boldsymbol{z}_+, \boldsymbol{z}_-$ such that

$$\boldsymbol{z}_+\boldsymbol{z}_- - \boldsymbol{z}_-\boldsymbol{z}_+ = \varepsilon(1 - \boldsymbol{z}_+\boldsymbol{z}_-)(1 - \boldsymbol{z}_-\boldsymbol{z}_+)$$

together with the normal ordering $\Omega(z_-^r z_+^s) = \boldsymbol{z}_-^r \boldsymbol{z}_+^s$. It is easy to see that $\tilde{\Omega}^{-1}(\boldsymbol{z}_+\boldsymbol{z}_-)$ will be an infinite expansion in ε.

We have shown that an ordered algebraic noncommutative geometry gives us a star product algebra. As mentioned in the introduction one can define a star algebra independently simply by specifying the functions $C_r \colon C^\omega(\mathcal{M}) \times C^\omega(\mathcal{M}) \mapsto C^\omega(\mathcal{M})$, and requiring that the star product defined by (4) is associative. We may now ask whether, given such an abstractly defined star product algebra, we can reconstruct an OANCG. For this we note the following:

- We construct the map $\dot{\pi} \colon \mathcal{A}^\star \mapsto \mathcal{A}^{\star 0} = \mathcal{A}^\star / \{\varepsilon \sim 0\}$. Normally $\mathcal{A}^{\star 0} = C^\omega(\mathcal{M})$, but $\mathcal{A}^0$ is an algebra of polynomials, therefore at best we can construct an OANCG $(\mathcal{A}_1, \Omega)$ so that $\mathcal{A}_1^\star$, the corresponding star algebra, is a subalgebra $\mathcal{A}_1^\star \subset \mathcal{A}^\star$.
- An OANCG gives more information than $\mathcal{A}^\star$, this is given by the immersion elements $\{\dot{x}_1, \ldots, \dot{x}_n\}$, $\dot{x}_i \in \mathcal{A}^\star$. These satisfy a set of immersion equations $\mathfrak{b}_s^0(\underline{x}) = 0$ where $x_i = \dot{\pi}(\dot{x}_i)$, which defines the immersion $\mathcal{M} \subset \mathbb{R}^n$.
- If $\mathcal{A}^\star$ is constructed from $(\mathcal{A}, \Omega)$ then $C_r(\boldsymbol{x}_i) \in \mathcal{A}^0$ where C_r is defined by (24). Thus we require that $C_r(\dot{x}_i)$ is a polynomial in x_k, where $C_r \colon \mathcal{A}^\star \mapsto \mathcal{A}^{\star 0}$ is given by $C_r(\dot{u})$ is the ε^r coefficient of $\dot{u}$.
- If $\mathcal{A}^\star$ is constructed from $(\mathcal{A}, \Omega)$ then $C_r(x_i, x_j) \in \mathcal{A}^0$, thus we require that $C_r(x_i, x_j)$ are polynomials in x_k.
- By considering the counter example above, in general it is possible only to construct an ε-infinite ANCG $\mathcal{A}^{(\infty)}$.

Given these conditions we can construct an OANCG. This is a precise statement (5).

Theorem 5. *Given a star algebra $\mathcal{A}^\star$, over a manifold $\mathcal{M}$, and a choice of immersion elements $\{\dot{x}_1, \ldots, \dot{x}_n\}$, $\dot{x}_i \in \mathcal{A}^\star$ such that the set $x_i = \pi(\dot{x}_i)$ define the immersion $\mathcal{M} = \{\mathfrak{b}^0(\underline{x}) = 0\} \subset \mathbb{R}^n$, and such that $C_r(x_j, x_k)$ and $C_r(\dot{x}_i)$ are polynomials in $(x_1, \ldots, x_n)$ for all r, i, j. Then there exists is a unique OANCG $(\mathcal{A}_1 = \mathcal{A}_1^{(\infty)}, \mathcal{M}, \Omega)$ for which $\mathcal{A}_1^\star$, the corresponding star algebra, is a subalgebra of $\mathcal{A}^\star$.*

Proof. Let $\mathcal{A}_1^\star$ be the subalgebra of $\mathcal{A}^\star$ generated by star products of $\{\dot{x}_1, \ldots, \dot{x}_n, \varepsilon\}$. Let $\mathcal{F} = \mathcal{F}^{(\infty)}$ be generated by $\{\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \varepsilon\}$. Let the function $\chi \colon \mathcal{F} \mapsto \mathcal{A}^\star$ be the algebraic homomorphism satisfying

$$\chi(\varepsilon) = \varepsilon\,,\ \chi(\mathfrak{x}_i) = \dot{x}_i\,,\ \chi(\mathfrak{u}\mathfrak{v}) = \chi(\mathfrak{u}) \star \chi(\mathfrak{v})$$

Now we show there is a map $\Phi \colon \mathcal{A}_1^\star \mapsto \mathcal{F}$ such that $\chi \circ \Phi = 1_{\mathcal{A}_1^\star}$.

Let $\Phi' \colon \mathcal{A}_1^\star \mapsto \mathcal{F}$ be any map such that $\pi \circ Q \circ \Phi' = \pi$, this can be constructed similar to proof of theorem 3. For $\dot{u} \in \mathcal{A}_1^\star$ let $\dot{u}_0 = \dot{u}$ and $\dot{u}_{n+1} = \dot{u}_n - \chi(\Phi'(\dot{u}_n))$. Thus $\dot{u}_n \in \mathcal{A}^\star$ and $\dot{u}_n = O(\varepsilon^n)$. Let $\Phi(\dot{u}) = \sum_{n=0}^\infty \Phi'(\dot{u}_n)$. We can say this converges in $\mathcal{F}^{(\infty)}$ and it is easy to show that $\chi\Phi(\dot{u}) = \dot{u}$.

Now let $\mathfrak{c}_{ij} = \mathfrak{x}_i\mathfrak{x}_j - \mathfrak{x}_j\mathfrak{x}_i - \Phi(\dot{x}_i \star \dot{x}_j - \dot{x}_j \star \dot{x}_i)$ and let $\mathfrak{b}_s = \Phi''(\mathfrak{b}_s^0) - \Phi\chi\Phi''(\mathfrak{b}_s^0)$ where $\Phi'' \colon \mathcal{F}^{C0} \mapsto \mathcal{F}$ is any map such that $Q^{C0} \circ \pi^F \circ \Phi'' = 1_{\mathcal{F}^{C0}}$.

The ordering is given by $\Omega = Q \circ \Phi$. This defines the map $\tilde{\Omega} \colon \mathcal{A}_1^\star \mapsto \mathcal{A}$.

Since $\tilde{\Omega}$ is a bijective homomorphism, then $\mathcal{A}_1$ is unique. ∎

2.4 Homomorphisms

Let $(\mathcal{A}_1, \varepsilon_1, \mathcal{M}_1)$ and $(\mathcal{A}_2, \varepsilon_2, \mathcal{M}_2)$ be two ANCG. We say that $\Psi\colon \mathcal{A}_1 \mapsto \mathcal{A}_2$ is a **homomorphism of ANCG** if Ψ is homomorphism of algebras and $\Psi(\varepsilon_1) = \varepsilon_2$. Let

$$\Psi^0\colon \mathcal{A}_1^0 \mapsto \mathcal{A}_2^0\,; \qquad \Psi^0 \circ \pi_1 = \pi_2 \circ \Psi \tag{27}$$

$$\Psi_\star\colon \mathcal{M}_2 \mapsto \mathcal{M}_1\,; \qquad \Psi^\star(u) = u \circ \Psi_\star\,, \qquad \forall u \in \mathcal{A}_1^0 \tag{28}$$

$$\Psi^0\colon C^\omega(\mathcal{M}_1) \mapsto C^\omega(\mathcal{M}_2)\,; \qquad \Psi^\star(u) = u \circ \Psi_\star\,, \qquad \forall u \in C^\omega(\mathcal{M}_1) \tag{29}$$

We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are **isomorphic** if the map Ψ is bijective.

Theorem 6. *The maps Ψ^0, $\Psi^\star$ and $\Psi_\star$ are well defined. The maps Ψ^0 and $\Psi^\star$ are homomorphisms and respect the Poisson structure:*

$$\Psi^\star\left(\{u, v\}\right) = \{\Psi^\star(u), \Psi^\star(v)\} \tag{30}$$

If Ψ^0 is surjective then $\Psi_\star$ is injective. If Ψ^0 is bijective then $\Psi_\star$ is bijective. Finally Ψ is bijective if and only if Ψ^0 is bijective

Proof. If $\boldsymbol{u} \in \mathcal{A}_1$ and $\pi_1(\boldsymbol{u}) = 0$ so $\boldsymbol{u} = \varepsilon_1 \boldsymbol{u}'$ for some $\boldsymbol{u}' \in \mathcal{A}_1$. Thus $\Psi(\boldsymbol{u}) = \Psi(\varepsilon_1 \boldsymbol{u}') = \Psi(\varepsilon_1)\Psi(\boldsymbol{u}') = \varepsilon_2 \Psi(\boldsymbol{u}')$ so $\pi_2(\Psi(u)) = 0$. Thus Ψ^0 is well defined, and clearly it is a homomorphism.

Let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n_1}\}$ and $\left\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{n_2}\right\}$ be the generators of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, and let $x_i = \pi_1(\boldsymbol{x}_i)$ and $y_i = \pi_2(\boldsymbol{y}_i)$. Given $u \in \mathcal{A}_1^0$ then $\Psi^0(u) \in \mathcal{A}_2^0$ is a polynomial in y_i. This includes the functions $\Psi^0(x_i)$. Since Ψ^0 is a homomorphism, then for any polynomial $f\colon \mathbb{R}^{n_1} \mapsto \mathbb{C}$ we have

$$f(\Psi^0(x_1), \ldots, \Psi^0(x_{n_1})) = \Psi^0(f(x_1, \ldots, x_{n_1})) \tag{31}$$

Given a point $p \in \mathcal{M}_2$, this has coordinates $(y_1(p), \ldots, y_{n_2}(p))$. The point $\Psi_\star(p)$ has coordinates $(\Psi^0(x_1)(p), \ldots, \Psi^0(x_{n_1})(p))$. Also for each immersions equation $\mathfrak{b}_s^0$ defining $\mathcal{A}_1$, given in (17) we have $\mathfrak{b}_s^0(\Psi^0(x_1), \ldots, \Psi^0(x_{n_1})) = 0$ from (31). So $\Psi_\star(p) \in \mathcal{M}_1$.

We define $\Psi^\star$ via $\Psi^\star(u) = u \circ \Psi_\star$ for $u \in C^\omega(\mathcal{M}_1)$. Since $\Psi^0(x^i)$ is a polynomial in y_i we can calculate all the partial derivative.

To show that the Poisson structure is preserved, given $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}_1$ we have

$$\begin{aligned}\Psi^0(\{\pi_1(\boldsymbol{u}), \pi_1(\boldsymbol{v})\}) &= \Psi^0(\pi_1(\tfrac{1}{i\varepsilon_1}[\boldsymbol{u}, \boldsymbol{v}])) = \pi_2(\Psi(\tfrac{1}{i\varepsilon_1})[\Psi(\boldsymbol{u}), \Psi(\boldsymbol{v})]) = \pi_2(\tfrac{1}{i\varepsilon_2}[\Psi(\boldsymbol{u}), \Psi(\boldsymbol{v})]) \\ &= \{\pi_2\Psi(\boldsymbol{u}), \pi_2\Psi(\boldsymbol{v})\} = \left\{\Psi^0(\pi_1(\boldsymbol{u})), \Psi^0(\pi_1(\boldsymbol{v}))\right\}\end{aligned}$$

And since the Poisson bracket is defined by a bi-vector then $\Psi^\star$ must also respect the Poisson structure.

Let Ψ^0 be surjective and $p, q \in \mathcal{M}_2$ such that $p \neq q$. Then there exists a function $u \in \mathcal{A}_2^0$ such that $u(p) \neq u(q)$. Since Ψ^0 is surjective there exists a function $v \in \mathcal{A}_1^0$ such that $\Psi^0(v) = u$. Thus $\Psi^0(v)(p) \neq \Psi^0(v)(q)$, giving $v(\Psi_\star(p)) \neq v(\Psi_\star(q))$, and hence $\Psi_\star(p) \neq \Psi_\star(q)$. Thus $\Psi_\star$ is injective.

If Ψ^0 is bijective then we have $\Psi^{0-1}\colon \mathcal{A}_2^0 \mapsto \mathcal{A}_1^0$ and hence $\Psi_\star^{-1}\colon \mathcal{M}_1 \mapsto \mathcal{M}_2$. Given $u \in \mathcal{A}_1^0$, $p \in \mathcal{M}_1$ we have

$$u(\Psi_\star^{-1}(\Psi_\star(p))) = \Psi^{0-1}(u)(\Psi_\star(p)) = \Psi^{0-1}(\Psi^\star(u))(p) = u(p)$$

Since this is for all u, then $\Psi_\star$ is bijective.

If Ψ is bijective then we can define Ψ^{0-1} via $\Psi^{0-1}\circ\pi_2=\pi_1\circ\Psi^{-1}$, and this satisfies $\Psi^{0-1}\Psi^0=1_{\mathcal{A}_1^0}$ and $\Psi^0\Psi^{0-1}=1_{\mathcal{A}_2^0}$.

If Ψ^0 is bijective then we first show that Ψ is injective. Let $\boldsymbol{u}\in\mathcal{A}_1$ such that $\Psi(\boldsymbol{u})=0$. Then $\pi_2\circ\Psi(\boldsymbol{u})=\Psi^0\circ\pi_1(\boldsymbol{u})=0$. Since Ψ^0 is injective $\pi_1(\boldsymbol{u})=0$. Thus $\boldsymbol{u}=\varepsilon_1\boldsymbol{u}_1$. So $0=\Psi(\boldsymbol{u})=\varepsilon_2\Psi(\boldsymbol{u}_1)$. So $\Psi(\boldsymbol{u}_1)=0$. Repeating this process shows $\boldsymbol{u}=0$.

If Ψ^0 is bijective then we show, by construction, that Ψ is surjective. Choose any ordering $\Omega_2\colon\mathcal{A}_2^0\mapsto\mathcal{A}_2$. Let $\boldsymbol{v}_n\in\mathcal{A}_2$, $n=0,1,\dots$ be defined inductively via

$$\begin{aligned}\boldsymbol{v}_0&=\Omega_2\circ\Psi^{0-1}\circ\pi_1(\boldsymbol{u})\\ \boldsymbol{v}_{n+1}&=\boldsymbol{v}_n+\varepsilon_2^n\Omega_2\circ\Psi^{0-1}\circ\pi_1\left(\frac{\boldsymbol{u}-\Psi(\boldsymbol{v}_n)}{\varepsilon_1^n}\right)\end{aligned}$$

Clearly $\boldsymbol{v}_{n+1}-\boldsymbol{v}_n=O(\varepsilon_1^n)$ so $\boldsymbol{v}_n$ converge to $\boldsymbol{v}_n\to\boldsymbol{v}\in\mathcal{A}_1$. Also $\boldsymbol{u}-\Psi(\boldsymbol{v}_n)=O(\varepsilon_1^n)$ so $\boldsymbol{u}=\Psi(\boldsymbol{v})$. □

If $(\mathcal{A}_1,\mathcal{M}_1,\Omega_1)$ and $(\mathcal{A}_2,\mathcal{M}_2,\Omega_2)$ are two OANCG then we say $\Psi\colon\mathcal{A}_2\mapsto\mathcal{A}_1$ is an **ANCG homeomorphism which respects the ordering** if

$$\Psi\circ\Omega_2=\Omega_1\circ\Psi^0 \tag{32}$$

This gives the following theorem.

Theorem 7. *If $\Psi\colon(\mathcal{A}_1,\Omega_1)\mapsto(\mathcal{A}_2,\Omega_2)$ is a ANCG homeomorphism which respects ordering then*

$$\Psi\circ\tilde{\Omega}_2=\tilde{\Omega}_1\circ\Psi^0 \tag{33}$$

and

$$\Psi^0(u\star_1 v)=\Psi^0(u)\star_2\Psi^0(v) \tag{34}$$

for $u,v\in\mathcal{A}_1^0$, where $\star_1$ and $\star_2$ are the star products corresponding to $\mathcal{A}_1^\star$ and $\mathcal{A}_2^\star$ respectively, and $\Psi^0:\mathcal{A}_1^\star\mapsto\mathcal{A}_2^\star$ is defined via linear extension, with $\Psi^0(\varepsilon_1)=\Psi^0(\varepsilon_2)$.

Proof. Since (32) is linear, then we can extend Ψ and Ω to $\mathcal{A}^\star$, thus (33). Now

$$\begin{aligned}\Psi^0(u\star_1 v)&=\Psi^0\circ\tilde{\Omega}_1^{-1}(\Omega_1(u)\Omega_1(v))=\tilde{\Omega}_2^{-1}\circ\Psi(\Omega_1(u)\Omega_1(v))=\tilde{\Omega}_2^{-1}(\Psi\circ\Omega_1(u)\Psi\circ\Omega_1(v))\\ &=\tilde{\Omega}_2^{-1}(\Omega_2\circ\Psi^0(u)\Omega_2\circ\Psi^0(v))=\Psi^0(u)\star_2\Psi^0(v)\end{aligned}$$

□

2.5 Representations and Trace

An additional structure that an ANCG may have is a representation or matrix representation. This is independent of whether or not an ordering is specified.

A **representations** of $\mathcal{A}$ over the Hilbert space $\mathcal{V}$ is a homomorphism

$$\varphi\colon\mathcal{A}\hookrightarrow L(\mathcal{V})\,;\qquad\varphi(\varepsilon)=\varepsilon_\infty\in\mathbb{R} \tag{35}$$

Here $L(\mathcal{V})$ is the space of linear (but not necessarily bounded) operators on $\mathcal{V}$. This representation is unitary if $\varphi(\boldsymbol{u}^\dagger)=\varphi(\boldsymbol{u})^\dagger$ where $\varphi(\boldsymbol{u})^\dagger$ is the adjoint with respect to the inner product on $\mathcal{V}$.

Clearly if $\mathcal{M}$ is compact and $\varepsilon_\infty = 0$ there is a natural unitary representation with $\mathcal{V} = L^2(\mathcal{M})$ as $\varphi(\boldsymbol{u})f = \pi(\boldsymbol{u})f$ with $f \in L^2(\mathcal{M})$. If $\varepsilon_\infty \neq 0$ then a prerequisite for the existence of a representation is that $\mathcal{A} = \mathcal{A}^{(\text{fin})}$. This is because the element $\sum_{r=0}^\infty \varepsilon^r r! \in \mathcal{A}^{(\infty)}$, and this does not have an image under φ.

We say there is a **matrix approximation** of $\mathcal{A}$ if there exists a sequence of $\varepsilon_N \in \mathbb{R}$ with $\varepsilon_N \neq 0$, and $\varepsilon_N \to 0$ as $N \to \infty$, such that

$$\varphi_N\colon \mathcal{A} \mapsto L(\mathbb{C}^N) = M_N(\mathbb{C})\,; \qquad \varphi_N(\varepsilon) = \varepsilon_N \tag{36}$$

Given an ANCG, it is not a trivial matter deciding whether there exists a unitary representation. In section 3 we give a number of examples of ANCGs with representation. Here we give an ANCG for a compact manifold, which does not possess a unitary matrix representation.

Counter example 2: This counter example is given by tensoring two copies of the noncommutative torus given in section 3.3. Let $\mathcal{M} = T^4$, and $\mathcal{A}$ be generated by $\{\varepsilon, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{u}_1^{-1}, \boldsymbol{u}_2^{-1}, \boldsymbol{v}_1^{-1}, \boldsymbol{v}_2^{-1}\}$. These obey

$$\boldsymbol{u}_1^r \boldsymbol{v}_1^s = e^{i\varepsilon rs} \boldsymbol{v}_1^s \boldsymbol{v}_1^r\,, \qquad \boldsymbol{u}_2^r \boldsymbol{v}_2^s = e^{i\varepsilon\alpha rs} \boldsymbol{v}_2^s \boldsymbol{v}_2^r\,, \qquad \boldsymbol{u}_i^\dagger = \boldsymbol{u}_i^{-1}\,, \qquad \boldsymbol{v}_i^\dagger = \boldsymbol{v}_i^{-1} \tag{37}$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is an irrational number, $i = 1, 2$ and all other commutators are zero.

Lemma 8. *The above ANCG does not have a unitary matrix representation.*

Proof. Let us assume we have a representation φ_N. Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $\varphi_N(\boldsymbol{u}_1)$. Then $\sum_i \lambda_i^r = \operatorname{tr}(\varphi_N(\boldsymbol{u}_1^r))$. This implies that there must exist an $r_1 > 0$ such that $\operatorname{tr}(\varphi_N(\boldsymbol{u}_1^{r_1})) \neq 0$. By looking at the trace of $\boldsymbol{v}_1 \boldsymbol{u}_1^r \boldsymbol{v}_1^{-1}$ we can show that $(1 - e^{ir\varepsilon_N})\operatorname{tr}(\varphi_N(\boldsymbol{u}_1^r)) = 0$. Hence $e^{ir_1\varepsilon_N} = 1$. Likewise $e^{ir_2\varepsilon_N\alpha} = 1$, for another integer $r_2 > 0$. This is impossible since α is not rational. $\square$

Given two ANCG, $\mathcal{A}_1$ and $\mathcal{A}_2$, with a homomorphism $\Psi\colon \mathcal{A}_1 \mapsto \mathcal{A}_2$. If both ANCG have matrix representations respectively given by $\varphi_N^{(1)}$ and $\varphi_N^{(2)}$, then this induces a matrix homomorphism Ψ_N for each N given by

$$\Psi_N\colon M_N(\mathbb{C}) \mapsto M_N(\mathbb{C})\,; \qquad \Psi_N \circ \varphi_N^{(1)} = \varphi_N^{(2)} \circ \Psi \tag{38}$$

Alternatively if only $\mathcal{A}_2$ has a representation $\varphi_N^{(2)}$ then we can induce a representation of $\mathcal{A}_1$

$$\varphi_N^{(1)} = \varphi_N^{(2)} \circ \Psi \tag{39}$$

However the representation generated in the way will not be surjective, unless Ψ is an isomorphism (see below). In this case we have the following trivial lemma.

Lemma 9. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic ANCG and $\varphi_N^{(2)}$ is a surjective matrix representation, then $\varphi_N^{(1)}$ is also a surjective matrix representation.*

If $(\mathcal{A}, \mathcal{M})$ has a matrix approximation we define the the **trace function** as the map

$$\operatorname{tr}_N\colon \mathcal{A} \mapsto \mathbb{C}\,; \qquad \operatorname{tr}_N(\boldsymbol{u}) = \tfrac{1}{N}\operatorname{tr}(\varphi_N(\boldsymbol{u})) \tag{40}$$

where $\operatorname{tr}\colon M_N(\mathbb{C}) \mapsto \mathbb{R}$ is the matrix trace. In general the matrix trace is dependent on the choice of matrix approximations however we do have the following trivial lemma

Lemma 10. *If $U_N \in GL_N(\mathbb{C})$ and φ_N is a matrix approximation of $(\mathcal{A}, \mathcal{M})$ then $\varphi'_N = U_N \varphi_N U_N^{-1}$ defines another matrix approximation of $\mathcal{A}$. In this case $\mathrm{tr}_N(\boldsymbol{u}) = \mathrm{tr}'_N(\boldsymbol{u})$.*

Theorem 11. *If $(\mathcal{A}, \mathcal{M})$ is symplectic, $\mathcal{M}$ is compact and tr_N exists for all N and $\lim_{N\to\infty} \mathrm{tr}_N(\boldsymbol{u})$ converges for all $\boldsymbol{u} \in \mathcal{A}$ then*

$$\lim_{N\to\infty} \mathrm{tr}_N(\boldsymbol{u}) = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \pi(\boldsymbol{u}) \omega^r \tag{41}$$

where $|\mathcal{M}| = \int_{\mathcal{M}} \omega$, and $\dim(\mathcal{M}) = 2r$.

Proof. Choose some ordering on $\mathcal{A}$. Define $\mathrm{tr}_0 \colon \mathcal{A}^0 \mapsto \mathbb{C}$ as $\mathrm{tr}_0(u) = \lim_{N\to\infty} \mathrm{tr}_N(\Omega(u))$. Since tr_0 is linear on $\mathcal{A}^0$ we can write

$$\mathrm{tr}_0(u) = \int_{\mathcal{M}} u W \omega^r \tag{42}$$

for some (distributional) weight function W on $\mathcal{M}$.

Let $(p_1, \ldots, p_r, q_1, \ldots, q_r)$ be conjugate coordinates on a patch of $\mathcal{M}$, and let $x_{2s} = p_s$ and $x_{2s+1} = q_s$. Let $\boldsymbol{u}_s \in \mathcal{A}$ and $u_s = \pi(\boldsymbol{u}_s)$, for $s = 1, \ldots, 2r$

$$\begin{aligned}
&\sum_{\sigma\in S_{2r}} \epsilon(\sigma) \left\{u_{\sigma(1)}, u_{\sigma(2)}\right\} \cdots \left\{u_{\sigma(2r-1)}, u_{\sigma(2r)}\right\} \omega^r \\
&\quad = r! \sum_{\sigma\in S_{2r}} \epsilon(\sigma) \sum_{i_1,\ldots i_r=1}^{r} \left(\frac{\partial u_{\sigma(1)}}{\partial p_{i_1}} \frac{\partial u_{\sigma(2)}}{\partial q_{i_1}} - \frac{\partial u_{\sigma(2)}}{\partial p_{i_1}} \frac{\partial u_{\sigma(1)}}{\partial q_{i_1}} \right) \cdots \times \\
&\qquad\qquad \left(\frac{\partial u_{\sigma(2r-1)}}{\partial p_{i_r}} \frac{\partial u_{\sigma(2r)}}{\partial q_{i_r}} - \frac{\partial u_{\sigma(2r)}}{\partial p_{i_r}} \frac{\partial u_{\sigma(2r-1)}}{\partial q_{i_r}} \right) dp_1 \wedge dq_1 \wedge \cdots \wedge dp_r \wedge dq_r \\
&\quad = 2^r r! \sum_{\sigma\in S_{2r}} \epsilon(\sigma) \sum_{i_1,\ldots i_r=1}^{r} \frac{\partial u_{\sigma(1)}}{\partial p_{i_1}} \frac{\partial u_{\sigma(2)}}{\partial q_{i_1}} \cdots \frac{\partial u_{\sigma(2r-1)}}{\partial p_{i_r}} \frac{\partial u_{\sigma(2r)}}{\partial q_{i_r}} dp_1 \wedge dq_1 \wedge \cdots \wedge dp_r \wedge dq_r \\
&\quad = 2^r r! \sum_{\sigma\in S_{2r}} \epsilon(\sigma) \sum_{\tau\in S_r} \frac{\partial u_{\sigma(1)}}{\partial p_{\tau(1)}} \frac{\partial u_{\sigma(2)}}{\partial q_{\tau(1)}} \cdots \frac{\partial u_{\sigma(2r-1)}}{\partial p_{\tau(r)}} \frac{\partial u_{\sigma(2r)}}{\partial q_{\tau(r)}} dp_1 \wedge dq_1 \wedge \cdots \wedge dp_r \wedge dq_r \\
&\quad = 2^r (r!)^2 \sum_{\sigma\in S_{2r}} \epsilon(\sigma) \frac{\partial u_{\sigma(1)}}{\partial p_1} \frac{\partial u_{\sigma(2)}}{\partial q_1} \cdots \frac{\partial u_{\sigma(2r-1)}}{\partial p_r} \frac{\partial u_{\sigma(2r)}}{\partial q_r} dp_1 \wedge dq_1 \wedge \cdots \wedge dp_r \wedge dq_r \\
&\quad = 2^r (r!)^2 \det_{ij} \left(\frac{\partial u_i}{\partial x_j} \right) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2r-1} \wedge dx_{2r} \\
&\quad = 2^r (r!)^2 du_1 \wedge du_2 \wedge \cdots \wedge du_{2r}
\end{aligned}$$

where S_r is the set of permutations, and $\epsilon(\sigma)$ is the signature of the permutation. However

$$\begin{aligned}
&\int_{\mathcal{M}} \omega^r \sum_{\sigma\in S_{2r}} \epsilon(\sigma) \left\{u_{\sigma(1)}, u_{\sigma(2)}\right\} \cdots \left\{u_{\sigma(2r-1)}, u_{\sigma(2r)}\right\} \\
&\quad = \lim_{N\to\infty} \sum_{\sigma\in S_{2r}} \epsilon(\sigma) \mathrm{tr}_N \left((i\varepsilon)^{-r} [\boldsymbol{u}_{\sigma(1)}, \boldsymbol{u}_{\sigma(2)}] \cdots [\boldsymbol{u}_{\sigma(2r-1)}, \boldsymbol{u}_{\sigma(2r)}] \right) \\
&\quad = \lim_{N\to\infty} 2^r (i\varepsilon_N)^{-r} \sum_{\sigma\in S_{2r}} \epsilon(\sigma) \mathrm{tr}_N \left(\boldsymbol{u}_{\sigma(1)} \boldsymbol{u}_{\sigma(2)} \cdots \boldsymbol{u}_{\sigma(2r)} \right) \\
&\quad = \lim_{N\to\infty} 2^r (i\varepsilon_N)^{-r} \sum_{\sigma\in S_{2r}} \epsilon(\sigma) \mathrm{tr}_N \left(\boldsymbol{u}_{\sigma(2)} \cdots \boldsymbol{u}_{\sigma(2r)} \boldsymbol{u}_{\sigma(1)} \right) = 0
\end{aligned}$$

since it is an odd permutation. Thus we have

$$\int_{\mathcal{M}} W du_1 \wedge du_2 \wedge \cdots \wedge du_{2r} = 0 \tag{43}$$

for all (algebraic) functions u_s on $\mathcal{M}$. Integration by parts gives

$$\int_{\mathcal{M}} u_1 dW \wedge du_2 \wedge \cdots \wedge du_{2r} = 0$$

By considering a sequence of u_1 we can let u_1 be the characteristic function on some subset $U_1 \in \mathcal{M}$. This implies

$$0 = \int_{U_1} dW \wedge du_2 \wedge \cdots \wedge du_{2r} = -\int_{\partial U_1} u_2 dW \wedge du_3 \wedge \cdots \wedge du_{2r}$$

Repeat this process until we are left with $\int_a^b dW = 0$, for two points $a, b \in \mathcal{M}$. This implies $W(a) = W(b)$. Thus W is a constant, whose value is given by $\mathrm{tr}_N(1) = 1$. □

If tr_N exists, we can define an inner product form:

$$\langle \bullet, \bullet \rangle_N : \mathcal{A} \times \mathcal{A} \mapsto \mathbb{C} \qquad \langle \boldsymbol{u}, \boldsymbol{v} \rangle_N = \mathrm{tr}_N(\boldsymbol{u}^\dagger \boldsymbol{v}) \tag{44}$$

This obeys $\langle [\boldsymbol{u}, \boldsymbol{v}], \boldsymbol{w} \rangle_N = \langle \boldsymbol{u}, [\boldsymbol{v}, \boldsymbol{w}] \rangle_N$.

We say tr_N is **analytic** if we can define a function $\mathrm{tr}_{\mathcal{A}} : \mathcal{A} \mapsto C^\omega(\mathbb{R})$, such that $\mathrm{tr}_{\mathcal{A}}(\boldsymbol{u})(\varepsilon_N) = \mathrm{tr}_N(\boldsymbol{u})$ for all N. For example the trace function on the noncommutative sphere and surface of rotation is analytic whilst the trace function on the noncommutative torus is not analytic, (see sections 3.4,3.5,3.4). If $\mathrm{tr}_{\mathcal{A}}$ exists then we can define the sesquilinear form

$$\langle \bullet, \bullet \rangle : \mathcal{A} \times \mathcal{A} \mapsto C^\omega(\mathbb{R}) \,; \qquad \langle \boldsymbol{u}, \boldsymbol{v} \rangle(\varepsilon) = \mathrm{tr}_{\mathcal{A}}(\boldsymbol{u}^\dagger \boldsymbol{u})(\varepsilon) \tag{45}$$

which satisfies $\langle \boldsymbol{u}, \boldsymbol{v} \rangle(\varepsilon_N) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle_N$. Although $\langle \bullet, \bullet \rangle_N$ is an inner product, $\langle \bullet, \bullet \rangle$ is, in general, not positive definite for all $\varepsilon \in \mathbb{R}$.

If $(\mathcal{A}, \Omega)$ is an OANCG then we say tr_N is compatible with Ω if $\mathrm{tr}_N(\Omega(u))$ is independent of N. One example is the sphere with the Wick-like ordering given by (98).

2.6 The Heisenberg Algebra and Coordinate Charts

For $r, s \in \mathbb{Z}$, $r \geq 1$, $s \geq 0$, we call the **Heisenberg algebra** $\mathcal{H}_{2r,s}$ the algebra generated by $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_r, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_r, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_s, \varepsilon\}$ with the only nonzero $\mathfrak{c}_{ij}$ given by $[\boldsymbol{p}_i, \boldsymbol{q}_j] = i\varepsilon\delta_{ij}$ and with no immersions equation $\mathfrak{b}_t$ so $D = 2r + s$. Clearly this is a ANCG for the manifold $\mathbb{R}^{2r+s}$. Thus $\mathcal{H}^0_{2r,s}$ is the algebra of polynomials on $\mathbb{R}^{2r+s}$. Each $\boldsymbol{y}_i$ is in the centre of $\mathcal{H}_{2r,s}$ so the corresponding symplectic leaves of $\mathbb{R}^{2r+s}$ are given by $y_i =$ constant. If $s = 0$ then we define $\mathcal{H}_{2r} = \mathcal{H}_{2r,s}$.

Two orderings on $\mathcal{H}_{2r,s}$ are commonly considered, the **Wick** ordering and the **normal** ordering. The Wick ordering is unique and is given by

$$\begin{aligned}\Omega_W(p_1^{i_1} \cdots p_r^{i_r} q_1^{j_1} \cdots q_r^{j_r} y_1^{k_1} \cdots y_s^{k_s}) = {} & \text{the correctly normalised sum of all symmetric} \\ & \text{permutations of } \boldsymbol{p}_1^{i_1} \cdots \boldsymbol{p}_r^{i_r} \boldsymbol{q}_1^{j_1} \cdots \boldsymbol{q}_r^{j_r} \boldsymbol{y}_1^{k_1} \cdots \boldsymbol{y}_s^{k_s}\end{aligned} \tag{46}$$

where correctly normalised means that $\pi \circ \Omega_W = 1_{\mathcal{H}^0_{2r,s}}$. The number of terms in the symmetric sum is given by

$$\frac{(i_1 + \cdots + i_r + j_1 + \cdots + j_r + k_1 + \cdots + k_s)!}{i_1! \cdots i_r! j_1! \cdots j_r! k_1! \cdots k_s!} \tag{47}$$

so we must divide by this quantity.

The normal ordering depends on the choices of an ordering on the generators of $\mathcal{H}_{2r,s}$. It is related to the time ordering in quantum field theory. The choice we will use here is to place $\boldsymbol{p}_i$ before $\boldsymbol{q}_i$ thus

$$\Omega_N(p_1^{i_1}\cdots p_r^{i_r}q_1^{j_1}\cdots q_r^{j_r}y_1^{k_1}\cdots y_s^{k_s}) = \boldsymbol{p}_1^{i_1}\cdots \boldsymbol{p}_r^{i_r}\boldsymbol{q}_1^{j_1}\cdots \boldsymbol{q}_r^{j_r}\boldsymbol{y}_1^{k_1}\cdots \boldsymbol{y}_s^{k_s} \tag{48}$$

For the Heisenberg plane $\mathcal{H}_2$ let the Wick basis elements $\boldsymbol{S}(a,b) = \Omega_W(p^a q^b)$ and the normal basis elements be $\boldsymbol{N}(a,b) = \boldsymbol{p}^a\boldsymbol{q}^b = \Omega_N(p^a q^b)$.

Theorem 12. *The Wick basis elements and normal basis elements are related by*

$$\boldsymbol{S}(a,b) = \sum_{r=0}^{\min(a,b)} \frac{(-\frac{1}{2}i\varepsilon)^r}{r!}\frac{a!}{(a-r)!}\frac{b!}{(b-r)!}\boldsymbol{N}(a,b) \tag{49}$$

$$\boldsymbol{N}(a,b) = \sum_{r=0}^{\min(a,b)} \frac{(\frac{1}{2}i\varepsilon)^r}{r!}\frac{a!}{(a-r)!}\frac{b!}{(b-r)!}\boldsymbol{S}(a,b) \tag{50}$$

The product of two basis elements are given by

$$\boldsymbol{N}(a,b)\boldsymbol{N}(c,d) = \sum_{r=0}^{\min(b,c)} \frac{(-i\varepsilon)^r}{r!}\frac{b!}{(b-r)!}\frac{c!}{(c-r)!}\boldsymbol{N}(a+c-r,b+d-r) \tag{51}$$

$$\boldsymbol{S}(a,b)\boldsymbol{S}(c,d) = \sum_{r=0} \frac{(i\varepsilon)^r}{r!}\boldsymbol{S}(a+c-r,b+d-r)\sum_{s=0}^{n}\frac{(-1)^{r-s}}{s!(r-s)!}\frac{a!}{(a-r+s)!}\frac{b!}{(b-s)!}\frac{c!}{(c-r+s)!}\frac{d!}{(d-s)!} \tag{52}$$

Proof. First note

$$\boldsymbol{p}\boldsymbol{S}(a,b) + \boldsymbol{S}(a,b)\boldsymbol{p} = 2\boldsymbol{S}(a+1,b)\,, \qquad \boldsymbol{q}\boldsymbol{S}(a,b) + \boldsymbol{S}(a,b)\boldsymbol{q} = 2\boldsymbol{S}(a,b+1)\,,$$
$$[\boldsymbol{p},\boldsymbol{S}(a,b)] = i\varepsilon b\boldsymbol{S}(a,b-1)\,, \qquad [\boldsymbol{q},\boldsymbol{S}(a,b)] = -i\varepsilon a\boldsymbol{S}(a-1,b)\,.$$

These are given in [8, appendex]. Also

$$\boldsymbol{p}\boldsymbol{N}(a,b) + \boldsymbol{N}(a,b)\boldsymbol{p} = 2\boldsymbol{N}(a+1,b) + \tfrac{1}{2}bi\varepsilon\boldsymbol{N}(a,b-1)$$

Thus (50) follows from induction on a, and (49) is its inverse.

Equation (51) follows from induction on b. For (52) expand $\boldsymbol{p}\boldsymbol{S}(a,b) + \boldsymbol{S}(a,b)\boldsymbol{p}$ and $\boldsymbol{q}\boldsymbol{S}(a,b) + \boldsymbol{S}(a,b)\boldsymbol{q}$. Then (52) follows from induction on a and b. □

Theorem 13. *The star product on $\mathcal{H}_{2r,s}$ with the Wick ordering is the Vey product.*

$$u \star_W v = \exp(\tfrac{1}{2}i\varepsilon\mathcal{P})(u,v) \tag{53}$$

where $\mathcal{P}$ is the Poisson operator given by $\mathcal{P}(u,v) = \{u,v\}$. That is

$$\mathcal{P} = \sum_i \left(\frac{\partial_1}{\partial p_i}\frac{\partial_2}{\partial q_i} - \frac{\partial_2}{\partial p_i}\frac{\partial_1}{\partial q_i}\right) \tag{54}$$

where the subscript $1,2$ refer to differentiation with respect to the first and second variable.

The star product on $\mathcal{H}_{2r,s}$ with the Normal ordering is

$$u \star_N v = \exp(-i\varepsilon\mathcal{N})(u,v) \qquad \textit{where} \qquad \mathcal{N} = \sum_i \frac{\partial_2}{\partial p_i}\frac{\partial_1}{\partial q_i} \tag{55}$$

Proof. To show this is true for $\mathcal{H}_2$ simply substitute $\boldsymbol{N}(a,b)$ into (55) and $\boldsymbol{S}(a,b)$ into (55) to obtain the corresponding product formulae. The results naturally extend for $\mathcal{H}_{2r,s}$. $\square$

In order to interpret $\mathcal{H}_{2r,s}$ as a coordinate basis we need to enlarge it to include certain analytic functions of the generators.

Let $\underline{a},\underline{b} \in \left(\mathbb{R}\cup\{\pm\infty\}\right)^{2r+s}$ such that $a_i < b_i$. Let $\mathcal{H}_{2r,s}(\underline{a},\underline{b})$ be the algebra generated by $\{f_i(\boldsymbol{p}_i), g_i(\boldsymbol{q}_i), h_i(\boldsymbol{y}_i), \varepsilon\}$ (with infinite sums of ε) where $f_i \in C^\omega(a_i,b_i)$, $g_i \in C^\omega(a_i+r,b_i+r)$ and $h_i \in C^\omega(a_i+2r,b_i+2r)$, and where $C^\omega(a_i,b_i)$ is the space of analytic functions on $\{x|a_i<x<b_i\}$. The following lemma shows that $\mathcal{H}_{2r,s}(\underline{a},\underline{b})$ is an algebra.

Lemma 14. *Every element of $\mathcal{H}_{2r,s}(\underline{a},\underline{b})$ may be written in the form*

$$\boldsymbol{u} = \sum_{t=0}^{\infty} \varepsilon^t \boldsymbol{u}_t \tag{56}$$

where $\boldsymbol{u}_t$ is a finite sum of terms of the form

$$f_1(\boldsymbol{p}_1)\cdots f_r(\boldsymbol{p}_r) g_1(\boldsymbol{q}_1)\cdots g_r(\boldsymbol{q}_r) h_1(\boldsymbol{y}_1)\cdots h_s(\boldsymbol{y}_s) \tag{57}$$

Proof. The formula for the normal star product Ω_N extends naturally to the elements of $\mathcal{H}^0_{2r,s}(\underline{a},\underline{b})$. Thus

$$g_i(\boldsymbol{q}_i)f_i(\boldsymbol{p}_i) = \sum_{r=0}^{\infty} \frac{(-i\varepsilon)^r}{r!} f_i^{(r)}(\boldsymbol{p}_i) g_i^{(r)}(\boldsymbol{q}_i) \tag{58}$$

Hence result. $\square$

Given $\mathcal{A}$ with $\dim(\mathcal{M}) = D = 2r+s$ we say there exists a **Heisenberg coordinate chart** of $\mathcal{A}$ if there exists an injective homeomorphism of ANCG $\Psi\colon \mathcal{A} \mapsto \mathcal{H}_{2r,s}(\underline{a},\underline{b})$.

Lemma 15. *If $\mathcal{A}$ is symplectic and $\mathcal{H}_{2r,s}$ is a coordinate chart for $\mathcal{A}^C$ then $\mathcal{H}_{2r}$ is a coordinate chart for $\mathcal{A}$. And the local immersions relations are*

$$\boldsymbol{y}_i = 0 \tag{59}$$

Proof. Trivial. $\square$

2.7 Quantum Groups

We can give many ANCGs a quantum group structure as a result of the two following theorems.

Theorem 16. *Let $\Psi : \mathcal{A}_1 \mapsto \mathcal{A}_2$ be a isomorphism of ANCG, and let $\mathcal{A}_1$ be a quantum group with coproduct Δ_1, counit ϵ_1 and antipode S_1, then $\mathcal{A}_2$ is also a quantum group with*

$$\Delta_2 = (\Psi\otimes\Psi)\circ\Delta_1\circ\Psi^{-1}\,, \qquad \epsilon_2 = \epsilon_1\circ\Psi^{-1}\,, \qquad S_2 = \Psi\circ S_1\circ\Psi^{-1}\,. \tag{60}$$

Proof. Simply go though all the axioms of a quantum group. $\square$

Theorem 17. *The Heisenberg ANCG is a Quantum Group.*

$$\begin{array}{ccc} \Delta(1) = 1\otimes 1\,, & \Delta(\varepsilon) = \varepsilon\otimes 1 + 1\otimes\varepsilon\,, & \Delta(\boldsymbol{x}) = \boldsymbol{x}\otimes 1 + 1\otimes \boldsymbol{x}\,, \\ \epsilon(1) = 1\,, & \epsilon(\varepsilon) = 0\,, & \epsilon(\boldsymbol{x}) = 0\,, \\ S(1) = 1\,, & S(\varepsilon) = -\varepsilon\,, & S(\boldsymbol{x}) = -\boldsymbol{x}\,, \\ & \forall \boldsymbol{x} \in \{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_r,\boldsymbol{q}_1,\ldots,\boldsymbol{q}_r,\boldsymbol{y}_1,\ldots,\boldsymbol{y}_s\} & \end{array} \tag{61}$$

Proof. Simply go though all the axioms of a quantum group. $\square$

We can use these theorems to give a quantum group structure to ANCG with coordinate charts. This will be used in the examples of the noncommutative torus and surface of rotation.

2.8 Generating a New ANCG by Use of a Homomorphism

Assume we have an ANCG $(\mathcal{A}_1, \mathcal{M}_1, \Omega_1)$, where $\mathcal{M}_1 \subset \mathbb{R}^{n_1}$, a second manifold $\mathcal{M}_2 \subset \mathbb{R}^{n_2}$, and an analytic bijective diffeomorphism $\Psi_\star : \mathcal{M}_2 \mapsto \mathcal{M}_1$. We can ask whether we can generate an OANCG $(\mathcal{A}_2, \mathcal{M}_2, \Omega_2)$ and a isomorphisms $\Psi\colon \mathcal{A}_1 \mapsto \mathcal{A}_2$ which respects ordering. This is important for the application later on when we wish to construct an OANCG on a general manifold. Unfortunately, in general, this is not possible. However, if $\star_1$ is differentiable, we use this to define the algebra $\mathcal{A}_2^\star$ via

$$\dot{u} \star_2 \dot{v} = \Psi^\star(\Psi^{\star-1}(\dot{u}) \star_1 \Psi^{\star-1}(\dot{v})) \tag{62}$$

We also define the immersions elements $\{\dot{y}_1, \ldots, \dot{y}_{n_2}\}$, $\dot{y}_i \in \mathcal{A}_2^\star$ as the coordinates of $\mathbb{R}^{n_2}$. However we cannot use theorem 5, because we can not guarantee that $C_r^{(2)}(\dot{y}_i, \dot{y}_j)$ is a polynomial.

Alternatively, if $\mathcal{A}_1$ has a matrix representation, we can use that. Let us assume that $(\mathcal{A}_2, \Omega_2)$ does exist, and let $\boldsymbol{x}_i \in \mathcal{A}_1$ and $\boldsymbol{y}_i \in \mathcal{A}_2$ be the corresponding bases. Then clearly $\varphi_N^{(2)} \circ \Omega_2 = \varphi_N^{(1)} \circ \Omega_1 \circ \Psi^{\star-1}$. So we have the matrix $Y_i^{(N)} = \varphi_N^{(2)}(\boldsymbol{y}_i) = \varphi_N^{(2)} \circ \Omega_2(y_i)$. Thus

$$Y_i^{(N)} = \varphi_N^{(1)} \circ \Omega_1 \circ \Psi^{\star-1}(y_i) \tag{63}$$

However we can define $Y_i^{(N)} \in M_N(\mathbb{C})$ using (63) even if $\mathcal{A}_2$ does not exist.

2.9 Geometric Properties of Surfaces

For many applications, especially gravity, we are interested in the geometric structure of $\mathcal{M}$, arising from a metric. Of course we are completely free to choose any metric on $\mathcal{M}$. However since we have the embedding : $\mathcal{M} \hookrightarrow \mathbb{R}^n$, we shall choose the metric $\mathcal{M}$ to be the pullback of the Euclidean metric on $\mathbb{R}^n$. Let $\sharp : T^\star\mathcal{M} \mapsto T\mathcal{M}$ be the metric dual given by $\xi(X) = g(\xi^\sharp, X)$. For this chapter we shall only consider two dimensional surfaces immersed in $\mathbb{R}^n$.

Theorem 18. *Let $\mathcal{M}$ be a surface embedded in $\mathbb{R}^n$ and let (p, q) be conjugate coordinates with $\{p, q\} = 1$ on a patch $U \subset \mathcal{M}$. The metric can be given solely in terms of the Poisson structure and the functions $x_i, p, q\colon U \mapsto \mathbb{R}$*

$$g(du^\sharp, dv^\sharp) = \frac{1}{C}\sum_i \{x_i, u\}\{x_i, v\} \tag{64}$$

where $C\colon U \mapsto \mathbb{R}$ is given by

$$C = \sum_{ij} \{p, x_i\}\{q, x_j\}\{x_j, x_i\} \tag{65}$$

Proof. This is basic manipulation

$$g = \sum_i dx_i \otimes dx_i = \sum_i \left(\left(\frac{\partial x_i}{\partial p}\right)^2 dp \otimes dp + \left(\frac{\partial x_i}{\partial q}\right)^2 dq \otimes dq + \frac{\partial x_i}{\partial p}\frac{\partial x_i}{\partial q}(dp \otimes dq + dq \otimes dp)\right)$$

Inverting this gives

$$\begin{aligned} g(du^\sharp, dv^\sharp) &= \frac{1}{C}\sum_i \left(\left(\frac{\partial x_i}{\partial q}\right)^2 \frac{\partial u}{\partial p}\frac{\partial v}{\partial p} + \left(\frac{\partial x_i}{\partial p}\right)^2 \frac{\partial u}{\partial q}\frac{\partial v}{\partial q} - \frac{\partial x_i}{\partial p}\frac{\partial x_i}{\partial q}\left(\frac{\partial u}{\partial p}\frac{\partial v}{\partial q} + \frac{\partial u}{\partial q}\frac{\partial v}{\partial p}\right)\right) \\ &= \frac{1}{C}\sum_i \{x_i, u\}\{x_i, v\} \end{aligned}$$

Here $C = \det(g)$ when written as a 2×2 matrix.

$$C = \sum_{ij} \left(\frac{\partial x_i}{\partial p} \frac{\partial x_i}{\partial p} \frac{\partial x_j}{\partial q} \frac{\partial x_j}{\partial q} - \frac{\partial x_i}{\partial p} \frac{\partial x_j}{\partial p} \frac{\partial x_i}{\partial q} \frac{\partial x_j}{\partial q} \right)$$

which gives (65) □

Let $\mathcal{M} \subset \mathbb{R}^n$ be a closed genus 0 symplectic surface, and let $\Psi_\star \colon \mathcal{M} \mapsto S^2$ be a bijective symplectic analytic diffeomorphism, and $\Psi^\star \colon C^\omega(S^2) \mapsto C^\omega(\mathcal{M})$ be the corresponding pullback map. Let (θ, ϕ) be the spherical coordinates on S^2, then $(p = \Psi^\star(\cos\theta), q = \Psi^\star(\phi))$ are conjugate coordinates on $\mathcal{M}$. However these coordinates are not defined for the whole of $\mathcal{M}$. More importantly, the noncommutative analogue of (p, q) do not have matrix representation. We can avoid this problem by setting $J_0 = \Psi^\star(\cos\theta)$, $J_1 = \Psi^\star(\sin\theta\cos\phi)$, and $J_2 = \Psi^\star(\sin\theta\sin\Phi)$.

The conformal factor C in (65) can now be written

$$C = \frac{1}{(1 - J_0^2)} \sum_{ij} \{x_j, x_i\} \{J_0, x_i\} (J_1 \{J_2, x_j\} - J_2 \{J_1, x_j\}) \tag{66}$$

The two above expression are examples of the following the theorem:

Theorem 19. *Let $\mathcal{M} \subset \mathbb{R}^n$ be a symplectic surface, and $u \colon \mathcal{M} \mapsto \mathbb{R}$ be a function that is derived from the metric on $\mathcal{M}$ and its embedding, using only differentiation. Then we can find an expression for u using only the Poisson bracket, the embedding functions $\{x_1, \ldots, x_n\}$ and the conjugate coordinates $\{p, q\}$. If $\mathcal{M}$ is topologically the sphere then we can replace $\{p, q\}$ with $\{J_0, J_1, J_2\}$.*

Proof. Take the expression for u and replace the metric with (65) or (66), and replace the derivatives using

$$\frac{\partial u}{\partial q} = \{p, u\} \ , \ \frac{\partial u}{\partial p} = -\{q, u\}$$

or

$$\frac{\partial u}{\partial q} = \{J_0, u\} \ , \ \frac{\partial u}{\partial p} = (1 - J_0^2)^{-1}(J_1 \{J_2, u\} - J_2 \{J_1, u\})$$

□

Examples of such functions include the curvature and Laplacian, which depend only on the metric, and the first and second fundamental forms, which depend on the metric and the embedding.

3 Examples

3.1 Heisenberg ANCG $\mathcal{H}_{2r,s}$

In section 2.6 we gave the details of the Heisenberg ANCG $\mathcal{H}_{2r,s}$, including the Wick and normal orderings and their corresponding star products. The Heisenberg algebra may be interpreted as the noncommutative Euclidean flat space $\mathbb{R}^{2r+s}$. Clearly the Heisenberg ANCG is its own coordinate chart. In section 2.7 we gave the quantum group based on $\mathcal{H}_{2r,s}$.

Because of the equation $[\boldsymbol{p}_i, \boldsymbol{q}_i] = i\varepsilon\delta_{ij}$, there do not exist any matrix representations of $\mathcal{H}_{2r,s}$. There do however exist many (topologically inequivalent) representations of $\mathcal{H}_{2r,s}$.

3.2 A Phase space $\mathcal{M} = T^\star S^2$

Non-relativistic quantum mechanics is obtained via the "quantisation" of phase space. In our language this means finding an ANCG $\mathcal{A}$ such that the corresponding manifold $\mathcal{M} = T^\star Q$ for some configuration space Q, and such that the inherited Poisson structure, is the canonical symplectic structure.

We give here an example corresponding to a free particle on a sphere, so that $Q = S^2$ and $\mathcal{M} = T^\star S^2$. Note that, in order to keep $\mathcal{A}$ algebraic, we require that we embed $T^\star S^2$ in $\mathbb{R}^8$, via the following embedding:

$$\begin{aligned} &x_1 = \sin\theta\,\cos\phi\,,\ x_2 = \sin\theta\,\sin\phi\,,\ x_3 = \cos\theta\,,\\ &x_4 = \cos\theta\,\cos\phi\,,\ x_5 = \cos\theta\,\sin\phi\,,\ x_6 = \sin\theta\,,\\ &x_7 = p_\theta\,, \qquad\qquad x_8 = p_\phi\,, \end{aligned} \tag{67}$$

where $(\theta, \phi, p_\theta, p_\phi)$ is a coordinate chart for $\mathcal{M}$, (θ, ϕ) are the standard spherical coordinates, and p_θ and p_ϕ there respective conjugate coordinates.

It is easy to show that the ANCG equivalent to the Heisenberg quantisation of $\mathcal{M} = T^\star S^2$ is generated by $\{\varepsilon, \boldsymbol{x}_1, \dots, \boldsymbol{x}_8\}$ with $\pi(\boldsymbol{x}_i) = x_i$. The commutation relations are

$$\begin{array}{llll} [\boldsymbol{x}_7, \boldsymbol{x}_1] = i\varepsilon\boldsymbol{x}_4\,, & [\boldsymbol{x}_7, \boldsymbol{x}_4] = -i\varepsilon\boldsymbol{x}_1\,, & [\boldsymbol{x}_8, \boldsymbol{x}_1] = -i\varepsilon\boldsymbol{x}_2\,, & [\boldsymbol{x}_8, \boldsymbol{x}_4] = i\varepsilon\boldsymbol{x}_5\,,\\ [\boldsymbol{x}_7, \boldsymbol{x}_2] = i\varepsilon\boldsymbol{x}_5\,, & [\boldsymbol{x}_7, \boldsymbol{x}_5] = -i\varepsilon\boldsymbol{x}_2\,, & [\boldsymbol{x}_8, \boldsymbol{x}_2] = i\varepsilon\boldsymbol{x}_1\,, & [\boldsymbol{x}_8, \boldsymbol{x}_5] = -i\varepsilon\boldsymbol{x}_4\,,\\ [\boldsymbol{x}_7, \boldsymbol{x}_3] = -i\varepsilon\boldsymbol{x}_6\,, & [\boldsymbol{x}_7, \boldsymbol{x}_6] = i\varepsilon\boldsymbol{x}_3\,, & [\boldsymbol{x}_8, \boldsymbol{x}_3] = 0\,, & [\boldsymbol{x}_8, \boldsymbol{x}_6] = 0\,,\\ [\boldsymbol{x}_i, \boldsymbol{x}_j] = 0 & \text{otherwise} & & \end{array} \tag{68}$$

and the immersion relations are

$$\begin{aligned} \boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 + \boldsymbol{x}_3^2 = 1\,, \qquad \boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 = \boldsymbol{x}_6^2\,,\\ \boldsymbol{x}_1\boldsymbol{x}_3 = \boldsymbol{x}_4\boldsymbol{x}_6\,, \qquad \boldsymbol{x}_2\boldsymbol{x}_3 = \boldsymbol{x}_5\boldsymbol{x}_6 \end{aligned} \tag{69}$$

from which all the other immersion equations can be derived.

To give the standard Hamiltonian for a free particle on a sphere it is necessary to enlarge $\mathcal{A}$ to include the generator $\boldsymbol{x}_9$ so that $x_9 = 1/\sin\theta$. Thus we must include the commutation relations $[\boldsymbol{x}_7, \boldsymbol{x}_9] = -i\varepsilon\boldsymbol{x}_3\boldsymbol{x}_9^2$, $[\boldsymbol{x}_9, \boldsymbol{x}_i] = 0$ for $i \neq 7$, and the immersion relation $\boldsymbol{x}_6\boldsymbol{x}_9 = 1$. Topologically this is the noncommutative version of the space $\mathcal{M} = T^\star(S^2 \setminus \{N, S\})$ where N, S are the two poles. The Hamiltonian is given by $\boldsymbol{H} = \frac{1}{2}\boldsymbol{x}_7^2 + \frac{1}{2}\boldsymbol{x}_8^2\boldsymbol{x}_9^2$. Of course this Hamiltonian is not unique and we can add any constant or any multiple of ε without effecting the classical dynamics.

The Schroedinger representation is given by $\mathcal{V} = L^2(S^2 \setminus \{N, S\})$, together with the inner product $\langle f, g\rangle = \int_{S^2} \overline{f} g \sin\theta\, d\theta\, d\phi$. The unbounded operators are given by $\varphi(\boldsymbol{x}_i) f = x_i f$, for $i = 1, \dots, 6$ and $\varphi(\boldsymbol{x}_7) f = i\varepsilon_\infty \partial_\theta f$ and $\varphi(\boldsymbol{x}_8) f = i\varepsilon_\infty \partial_\phi f$.

A Heisenberg coordinate chart is given by

$$\Psi\colon \mathcal{A} \mapsto \mathcal{H}_4\left(\begin{pmatrix} -\infty\\ -\infty\\ -\infty\\ -\infty \end{pmatrix}, \begin{pmatrix} \infty\\ \infty\\ \infty\\ \infty \end{pmatrix}\right) \qquad \text{with coordinates} \quad (\boldsymbol{\theta}, \boldsymbol{\phi}, \boldsymbol{p_\theta}, \boldsymbol{p_\phi}) \text{ where } [\boldsymbol{p_\theta}, \boldsymbol{\theta}] = [\boldsymbol{p_\phi}, \boldsymbol{\phi}] = i\varepsilon$$

The immersion elements $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_8\}$ are given by (67), but replacing the unbolded with the bolded symbols.

3.3 Torus or Manin plane

For historical reasons the noncommutative torus is often called the Manin plane or Weyl algebra. To place it in our language, $\mathcal{A}_{T^2}$ is generated by $\{\varepsilon, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$ with $\boldsymbol{x}_1 = \frac{1}{2}(\boldsymbol{u} + \boldsymbol{u}^{-1})$,

$\boldsymbol{x}_2 = \frac{1}{2i}(\boldsymbol{u} - \boldsymbol{u}^{-1})$, $\boldsymbol{x}_3 = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{v}^{-1})$, $\boldsymbol{x}_4 = \frac{1}{2i}(\boldsymbol{v} - \boldsymbol{v}^{-1})$. The relations are given by

$$\boldsymbol{u}^r\boldsymbol{v}^s = e^{i\varepsilon rs}\boldsymbol{v}^r\boldsymbol{v}^s\,, \qquad \boldsymbol{u}^{-1}\boldsymbol{u} = \boldsymbol{u}\boldsymbol{u}^{-1} = \boldsymbol{v}^{-1}\boldsymbol{v} = \boldsymbol{v}\boldsymbol{v}^{-1} = 1\,, \qquad \boldsymbol{u}^\dagger = \boldsymbol{u}^{-1}\,, \qquad \boldsymbol{v}^\dagger = \boldsymbol{v}^{-1} \tag{70}$$

where $r, s = \pm 1$. We can show that the first equation above is true for all $r, s \in \mathbb{Z}$.

There is a coordinate systems for the noncommutative torus given by

$$\Psi_{T^2} \colon \mathcal{A} \mapsto \mathcal{H}_2\left(\binom{-\infty}{-\infty}, \binom{\infty}{\infty}\right)\;; \qquad \Psi_{T^2}(\boldsymbol{u}) = e^{i\varepsilon \boldsymbol{p}}\,, \qquad \Psi_{T^2}(\boldsymbol{v}) = e^{i\varepsilon \boldsymbol{q}} \tag{71}$$

To get the Vey product we must use central ordering (theorem 20 below) given by $\Omega_V(u^r v^s) = \boldsymbol{u}^r\boldsymbol{v}^s e^{-irs\varepsilon/2}$.

A normal ordering is given by $\Omega_N(u^r v^s) = \boldsymbol{u}^r\boldsymbol{v}^s$. This produces the following star product

$$f \star_N g = \exp\left(-i\varepsilon\frac{\partial_2}{\partial u}\frac{\partial_1}{\partial v}\right)(f, g) \tag{72}$$

There is a matrix representation of $\mathcal{A}_{T^2}$ given with respect to the basis $\{|0\rangle, \ldots, |N-1\rangle\}$

$$\varphi_N(\boldsymbol{u})|r\rangle = e^{ir\varepsilon_N}|r\rangle\,, \qquad \varphi_N(\boldsymbol{v})|0\rangle = |N-1\rangle\,,\ \varphi_N(\boldsymbol{v})|r\rangle = |r-1\rangle\,,\ r = 1, \ldots, N-1 \tag{73}$$

where $\varepsilon_N = 1/N$. The trace map is therefore given by

$$\mathrm{tr}_N(\boldsymbol{u}^r\boldsymbol{v}^s) = \delta(r \bmod N)\,\delta(s \bmod N) \tag{74}$$

Therefore $\lim_{N\to\infty} \mathrm{tr}_N(\boldsymbol{y})$ exists for all $\boldsymbol{y} \in \mathcal{A}_{T^2}$, and theorem 11 applies. However tr_N is not analytic.

There is a Quantum Group structure for $\mathcal{A}_{T^2}$, suggested by section 2.7, given by

$$\begin{array}{llll} \Delta(1) = 1 \otimes 1\,, & \Delta(e^{i\varepsilon}) = e^{i\varepsilon} \otimes e^{i\varepsilon}, & \Delta(\boldsymbol{u}^r) = \boldsymbol{u}^r \otimes \boldsymbol{u}^r\,, & \Delta(\boldsymbol{v}^r) = \boldsymbol{v}^r \otimes \boldsymbol{v}^r \\ \epsilon(1) = 1\,, & \epsilon(e^{i\varepsilon}) = 1\,, & \epsilon(\boldsymbol{u}^r) = 1\,, & \epsilon(\boldsymbol{v}^r) = 1\,, \\ S(1) = 1\,, & S(e^{i\varepsilon}) = e^{-i\varepsilon}\,, & S(\boldsymbol{u}^r) = \boldsymbol{u}^{-r}\,, & S(\boldsymbol{v}^r) = \boldsymbol{v}^{-r}\,, \end{array} \tag{75}$$

for all $r \in \mathbb{Z}$.

3.4 Surfaces of Rotation

These were first introduced in [9] then expanded in [10]. The ANCG, $\mathcal{A}_\rho$, are generated by $\{\varepsilon, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ and defined with respect to a polynomial function $\rho \colon \mathbb{R}^2 \mapsto \mathbb{R}$. The quotient relations are given by

$$\begin{gathered} [\boldsymbol{X}_0, \boldsymbol{X}_+] = \varepsilon\boldsymbol{X}_+\,, \qquad [\boldsymbol{X}_0, \boldsymbol{X}_-] = -\varepsilon\boldsymbol{X}_-\,, \qquad [\boldsymbol{X}_+, \boldsymbol{X}_-] = \rho(\boldsymbol{X}_0 - \varepsilon/2, \varepsilon) - \rho(\boldsymbol{X}_0 + \varepsilon/2, \varepsilon)\,, \\ \boldsymbol{X}_+\boldsymbol{X}_- + \boldsymbol{X}_-\boldsymbol{X}_+ = \rho(\boldsymbol{X}_0 - \varepsilon/2, \varepsilon) + \rho(\boldsymbol{X}_0 + \varepsilon/2, \varepsilon) \end{gathered} \tag{76}$$

where $\boldsymbol{x}_1 = \frac{1}{2}(\boldsymbol{X}_+ + \boldsymbol{X}_-)$, $x_2 = \frac{1}{2i}(\boldsymbol{X}_+ - \boldsymbol{X}_-)$, $\boldsymbol{x}_3 = \boldsymbol{X}_0$.

The topology of the corresponding $\mathcal{M}$ depends on the shape of the curve $y(z) = \rho(z, 0)$. If we let $I_\rho(0) = \{z \in \mathbb{R} | \rho(z, 0) \geq 0\}$ then $I_\rho(0)$ is the union of intervals. Assuming that $\rho(z, 0) \neq 0$ on the interior of $I_\rho(0)$ then each bounded interval in $I_\rho(0)$ corresponds to a disjoint submanifold topologically equivalent to the sphere. If one of the intervals is either $\{z | -\infty < z < z_{\mathrm{hi}}\}$ or $\{z | z_{\mathrm{lo}} < z < \infty\}$ then the corresponding submanifold is topologically the disc. Finally if $I_\rho(0) = \mathbb{R}$ then $\mathcal{M}$ is topologically a cylinder. For a ρ with several maxima there may be

several intervals in $I_\rho(0)$, and therefore $\mathcal{M}$ is disconnected. Replacing $\rho(z,\varepsilon) \to \rho(z,\varepsilon) + C$ may change the topology to $\mathcal{M}$. This is analysed in [10].

If $\rho(z,\varepsilon) = z^2$ then $\mathcal{M}$ is not a manifold, but an algebraic variety. However much of the analysis is still valid in this case.

For $\varepsilon_0 \in \mathbb{R}$, $\varepsilon_0 \geq 0$ let $I_\rho(\varepsilon_0) = \{z \in \mathbb{R} | \rho(z,\varepsilon_0) > 0\}$. If there exists $\varepsilon_N > 0$ such that $I_\rho(\varepsilon_N)$ is a bounded interval given by $I_\rho(\varepsilon_N) = \{z | z_{\rm lo}(\varepsilon_N) < z < z_{\rm hi}(\varepsilon_N)\}$ where $N\varepsilon_N = z_{\rm hi}(\varepsilon_N) - z_{\rm lo}(\varepsilon_N)$, then there is a $M_N(\mathbb{C})$ representation of $\mathcal{A}_\rho$ given by

$$\begin{aligned}
\varphi_N(\boldsymbol{X}_0)|r\rangle &= \left(z_{\rm lo}(\varepsilon_N) + (r+\tfrac{1}{2})\varepsilon_N\right)|r\rangle \\
\varphi_N(\boldsymbol{X}_+)|r\rangle &= \rho(z_{\rm lo}(\varepsilon_N) + (r+1)\varepsilon_N, \varepsilon_N)^{1/2}|r+1\rangle \\
\varphi_N(\boldsymbol{X}_-)|r\rangle &= \rho(z_{\rm lo}(\varepsilon_N) + \varepsilon_N, \varepsilon_N)^{1/2}|r-1\rangle
\end{aligned} \tag{77}$$

If for some $\varepsilon_\infty > 0$, $I_\rho(\varepsilon_\infty)$ is an unbounded interval then there are infinite dimensional representations of $\mathcal{A}_\rho$. It is easy to see that if $I_\rho(\varepsilon_0)$ is a bounded interval for all $\varepsilon_0 > 0$ then the trace map is defined, and furthermore it is analytic.

If $I_\rho(\varepsilon_0)$ is a single interval, possibly unbounded, for all $\varepsilon_0 > 0$, then the coordinate system is given by

$$\begin{array}{cc}
\Psi\colon \mathcal{A}_\rho \mapsto \mathcal{H}_2\left(\binom{z_{\rm lo}}{-\infty}, \binom{z_{\rm hi}}{\infty}\right) ; & \Psi(\boldsymbol{X}_0) = \boldsymbol{p}\,, \\
\Psi(\boldsymbol{X}_+) = e^{i\boldsymbol{q}}(\rho(\boldsymbol{p}+\tfrac{1}{2}\varepsilon,\varepsilon))^{1/2}\,, & \Psi(\boldsymbol{X}_-) = e^{-i\boldsymbol{q}}(\rho(\boldsymbol{p}-\tfrac{1}{2}\varepsilon,\varepsilon))^{1/2}
\end{array} \tag{78}$$

If $I_\rho(\varepsilon_\infty)$ is an unbounded interval we can replace $z_{\rm lo}$ with $-\infty$ or $z_{\rm hi}$ with $+\infty$ or both.

Let $U_\rho = \{(z,\varepsilon_0) \in \mathbb{R}^2 \,|\, \rho(z,\varepsilon_0) > 0\}$. It is useful to enlarge $\mathcal{A}$ to the set

$$\mathcal{A}_\rho = \left\{ \sum_{r=0}^{\text{finite}} \boldsymbol{X}_+ f_r(\boldsymbol{X}_0,\varepsilon) + \sum_{r=0}^{\text{finite}} \boldsymbol{X}_- f_{-r}(\boldsymbol{X}_0,\varepsilon) \right\} \tag{79}$$

where $f_r\colon U \mapsto \mathbb{C}$ is C^ω on the interior of U_ρ. From (76) we have that

$$f(\boldsymbol{X}_0,\varepsilon)\boldsymbol{X}_\pm = \boldsymbol{X}_\pm f(\boldsymbol{X}_0 \pm \varepsilon, \varepsilon). \tag{80}$$

Because of this extension, we can talk about $\mathcal{A}_\rho$ even when $\rho\colon U_\rho \mapsto \mathbb{R}$ is bounded and C^ω on the interior of U_ρ.

As well as the homeomorphism giving the coordinate system, there are isomorphisms between certain topologically equivalent noncommutative surfaces of rotation. For example let $\mathcal{A}_{\rho_1}$ and $\mathcal{A}_{\rho_2}$ be noncommutative surfaces of rotation with generators $\varepsilon_1, \boldsymbol{X}_0, \boldsymbol{X}_+, \boldsymbol{X}_-$ and $\varepsilon_2, \boldsymbol{Y}_0, \boldsymbol{Y}_+, \boldsymbol{Y}_-$ respectively, such that ρ_1, ρ_2 independent to ε, and both $I_{\rho_1} = \{z | z^1_{\rm lo} < z < z^1_{\rm hi}\}$ and $I_{\rho_2} = \{z | z^2_{\rm lo} < z < z^2_{\rm hi}\}$ are bounded then

$$\begin{array}{l}
\Psi\colon \mathcal{A}_{\rho_1} \mapsto \mathcal{A}_{\rho_2}\,; \qquad \Psi(\varepsilon_1) = \varepsilon_2\,, \qquad \Psi(\boldsymbol{X}_0) = K(\boldsymbol{Y}_0 - z^2_{\rm lo}) + z^1_{\rm lo}\,, \\
\Psi(\boldsymbol{X}_+) = \boldsymbol{Y}_+ \left(\dfrac{\rho_1(K(\boldsymbol{Y}_0 - z^2_{\rm lo}) + z^1_{\rm lo} + \frac{1}{2}\varepsilon_2)}{\rho_2(\boldsymbol{Y}_0 + \frac{1}{2}\varepsilon_2)} \right)^{1/2}, \qquad \Psi(\boldsymbol{X}_-) = \Psi(\boldsymbol{X}_+)^\dagger
\end{array} \tag{81}$$

where

$$K = \frac{(z^1_{\rm hi} - z^1_{\rm lo})}{(z^2_{\rm hi} - z^2_{\rm lo})}$$

One possible ordering is the normal ordering given by

$$\Omega_N(X^r_\pm f(X_0)) = \boldsymbol{X}^r_\pm f(\boldsymbol{X}_0) \tag{82}$$

This ordering does not correspond to a differential star product with $C_1 : \mathcal{A}^\star \times \mathcal{A}^\star \mapsto \mathcal{A}^\star$ a first order operator. To see this we note that

$$C_1(X_+^r, X_-^r) = r\rho_\varepsilon(X_0,0)\rho(X_0,0)^{r-1} + 2^{-2r}(2r)!(n!)^{-1}\rho_p(X_0,0)\rho(X_0,0)^{r-1}$$

where ρ_p and ρ_ε are the partial differentiation of $\rho(p,\varepsilon)$ with respect to the the first and second arguments respectively.

In order to get the Vey product we need the **central ordering** which is defined with respect to the coordinates $(\boldsymbol{p},\boldsymbol{q})$. If Ψ is the coordinate homomorphism (78) then

$$\Omega_C(u) = \boldsymbol{u}\,, \text{ where } \Psi^0(u) = e^{irq}f(p) \text{ and } \Psi(\boldsymbol{u}) = e^{ir\boldsymbol{q}}f(\boldsymbol{p}+r\varepsilon/2) \tag{83}$$

In terms of the elements of $\mathcal{A}_\rho$ we can show that

$$\begin{aligned}\Omega_C(X_+^r f(X_0)) &= \boldsymbol{X}_+^r\left(\frac{(\rho(X_0+\frac{1}{2}r\varepsilon,\varepsilon))^{r-1}}{\rho(\boldsymbol{X}_0+\frac{3}{2}\varepsilon,\varepsilon)\rho(\boldsymbol{X}_0+\frac{5}{2}\varepsilon,\varepsilon)\cdots\rho(\boldsymbol{X}_0+\frac{2r-1}{2}\varepsilon,\varepsilon)}\right)^{1/2} f(\boldsymbol{X}_0+\tfrac{1}{2}r\varepsilon)\\ \Omega_C(X_-^r f(X_0)) &= \boldsymbol{X}_-^r\left(\frac{(\rho(X_0-\frac{1}{2}r\varepsilon,\varepsilon))^{r-1}}{\rho(\boldsymbol{X}_0-\frac{3}{2}\varepsilon,\varepsilon)\rho(\boldsymbol{X}_0-\frac{5}{2}\varepsilon,\varepsilon)\cdots\rho(\boldsymbol{X}_0-\frac{2r-1}{2}\varepsilon,\varepsilon)}\right)^{1/2} f(\boldsymbol{X}_0-\tfrac{1}{2}r\varepsilon)\end{aligned} \tag{84}$$

Theorem 20. *The central ordering is compatible with the Wick ordering under the Heisenberg coordinate homeomorphism:*

$$\Psi\circ\Omega_C = \Omega_W\circ\Psi^0 \tag{85}$$

where Ψ is given by (78) and Ω_W by (46). Hence the central ordering gives the Vey product.

Proof. From (50) we have

$$\begin{aligned}\Omega_W(e^{ibq}p^a) &= \sum_{s=0}^\infty \boldsymbol{S}(a,s)\frac{(bi)^s}{s!} = \sum_{s=0}^\infty\sum_{r=0}^a \frac{b^s i^{s-r}(\frac{-\varepsilon}{2})^r a!}{(a-r)!r!(s-r)!}\boldsymbol{p}^{a-r}\boldsymbol{q}^{s-r}\\ &= \sum_{r=0}^\infty\sum_{t=0}^a \frac{(bi)^t}{t!}\frac{b^r(\frac{-\varepsilon}{2})^r a!}{(a-r)!r!}\boldsymbol{p}^{a-r}\boldsymbol{q}^t = (\boldsymbol{p}-b\varepsilon/2)^a e^{i\boldsymbol{q}}\end{aligned}$$

Hence (83). Using theorem 7 shows that the star product must be Vey.

We can also prove that the central ordering gives the Vey product directly. Let $u = e^{inq}f(p)$ and $v = e^{imq}g(p)$ then from the definition of the Vey product we have

$$\begin{aligned}u\star v &= \sum_{s=0}^\infty \frac{(i\varepsilon/2)^s}{s!}P^s(F,G)\\ &= \sum_{s=0}^\infty \frac{(i\varepsilon/2)^s}{s!}\left(\frac{\partial_1}{\partial p}\frac{\partial_2}{\partial q} - \frac{\partial_1}{\partial p}\frac{\partial_2}{\partial q}\right)^s (u,v)\\ &= \sum_{t=0}^\infty \frac{(i\varepsilon/2)^t}{t!}\left(\frac{\partial_1}{\partial p}\frac{\partial_2}{\partial q}\right)^t \sum_{r=0}^\infty \frac{(i\varepsilon/2)^r}{r!}\left(\frac{\partial_2}{\partial p}\frac{\partial_1}{\partial q}\right)^r (u,v)\\ &= e^{iq(n+m)}f(p+m\varepsilon/2)g(q-n\varepsilon/2)\end{aligned}$$

where $P(u,v)=\{u,v\}$ and $(\partial_1/\partial p)$ refers to differentiating with respect to u and $(\partial_2/\partial p)$ refers to differentiating with respect to v. Thus

$$\begin{aligned}\tilde{\Omega}_C(u\star v) &= e^{i\boldsymbol{q}(n+m)}f(\boldsymbol{p}+m\varepsilon+n\varepsilon/2)g(q+m\varepsilon/2)\\ &= e^{i\boldsymbol{q}n}f(\boldsymbol{p}+n\varepsilon/2)e^{i\boldsymbol{q}m}g(q+m\varepsilon/2)\\ &= \tilde{\Omega}_C(u)\tilde{\Omega}_C(v)\end{aligned}$$

□

In general we can only give the quantum group structure in terms of a formal expansion. By directly applying (61) on (78), we get

$$\Delta(\varepsilon)=0\,,\qquad \Delta(\boldsymbol{X}_0)=1\otimes\boldsymbol{X}_0+\boldsymbol{X}_0\otimes 1\,, \tag{86}$$

$$\Delta(X_+)=\sum_{a,b=0}^{\infty}\sum_{s=0}^{a}\sum_{t=0}^{b}\frac{\alpha_{ab}a!b!}{s!(a-s)!t!(b-t)!}\varepsilon^s\boldsymbol{X}_+\rho(\boldsymbol{X}_0+\tfrac{1}{2}\varepsilon,\varepsilon)^{-1/2}\boldsymbol{X}_0^t\otimes\varepsilon^{a-s}\boldsymbol{X}_+\rho(\boldsymbol{X}_0+\tfrac{1}{2}\varepsilon,\varepsilon)^{-1/2}\boldsymbol{X}_0^{b-t}\,,$$

$$\Delta(X_-)=\sum_{a,b=0}^{\infty}\sum_{s=0}^{a}\sum_{t=0}^{b}\frac{\alpha_{ab}a!b!}{s!(a-s)!t!(b-t)!}\varepsilon^s\rho(\boldsymbol{X}_0+\tfrac{1}{2}\varepsilon,\varepsilon)^{-1/2}\boldsymbol{X}_0^t\boldsymbol{X}_-\otimes\varepsilon^{a-s}\rho(\boldsymbol{X}_0+\tfrac{1}{2}\varepsilon,\varepsilon)^{-1/2}\boldsymbol{X}_0^{b-t}\boldsymbol{X}_-\,,$$

$$\begin{aligned}&\epsilon(\varepsilon)=0\,,\qquad \epsilon(\boldsymbol{X}_0)=0\,,\qquad \epsilon(\boldsymbol{X}_+)=\rho(0,0)\,,\qquad \epsilon(\boldsymbol{X}_-)=\rho(0,0)\,,\\ &S(\varepsilon)=-\varepsilon\,,\qquad S(\boldsymbol{X}_0)=-\boldsymbol{X}_0\,,\\ &S(\boldsymbol{X}_+)=\boldsymbol{X}_-\rho(-\boldsymbol{X}_0+\tfrac{1}{2}\varepsilon,-\varepsilon)^{1/2}\rho(\boldsymbol{X}_0-\tfrac{1}{2}\varepsilon,\varepsilon)^{-1/2}\,,\\ &S(\boldsymbol{X}_-)=\rho(-\boldsymbol{X}_0+\tfrac{1}{2}\varepsilon,-\varepsilon)^{1/2}\rho(\boldsymbol{X}_0-\tfrac{1}{2}\varepsilon,\varepsilon)^{-1/2}\boldsymbol{X}_+\,,\end{aligned}$$

where the Taylor expansions of $\rho^{1/2}$ is given by

$$\rho(u+\tfrac{1}{2}\varepsilon,\varepsilon)^{1/2}=\sum_{a,b=0}^{\infty}\alpha_{ab}\varepsilon^a u^b$$

Note these simplifies a little if $\rho(u,\varepsilon)=\rho(-u,-\varepsilon)$. Clearly for the image of Δ in (86) to be a polynomial requires that $\rho(u+\frac{1}{2}\varepsilon,\varepsilon)^{1/2}$ is a polynomial. This implies that $\mathcal{M}_\rho$ is topologically the cylinder. (Thus excluding the sphere.) Examples of such ρ include $\rho(u,\varepsilon)=1$ and $\rho(u,\varepsilon)=(u^2+1)^2$,

3.5 The Sphere

The noncommutative sphere has been studied by many authors [13, 8, 7, 2]. It is an example of a noncommutative surface of rotation with

$$\rho(z,\varepsilon)=R^2-z^2+\varepsilon^2/4 \tag{87}$$

where $R\in\mathbb{R}$ gives the radius of the embedded sphere. By looking at the commutation relations part of (76) we see that $\{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3\}$ obey the commutation relations of the Lie algebra $su(2)$, given by $[\boldsymbol{x}_i,\boldsymbol{x}_j]=i\varepsilon\,\epsilon_{ijk}\boldsymbol{x}_k$. The immersion equation of (76) gives the Casimir $\boldsymbol{x}_1^2+\boldsymbol{x}_2^2+\boldsymbol{x}_3^2=R^2$. As a result [7] both $\mathcal{A}_{S^2}$ and $\mathcal{A}_{S^2}^C$ are infinite dimensional representations of $su(2)$, with $\mathcal{A}_{S^2}^C$ being the enveloping algebra.

All the results for noncommutative surface of rotation now carry over, including the Vey or central ordering. The finite dimensional unitary representation of $\mathcal{A}_{S^2}$ given by (77) reduces to the standard representations of $su(2)$

$$\begin{aligned} \varphi_N(\varepsilon) &= \varepsilon_N = 2R(N^2-1)^{-1/2} \\ \varphi_N(\boldsymbol{X}_0)|m\rangle &= \varepsilon_N(m - \tfrac{N-1}{2})|m\rangle \\ \varphi_N(\boldsymbol{X}_+)|m\rangle &= \varepsilon_N(N-m-1)^{1/2}(m+1)^{1/2}|m+1\rangle \\ \varphi_N(\boldsymbol{X}_-)|m\rangle &= \varepsilon_N(N-m)^{1/2}(m)^{1/2}|m-1\rangle \end{aligned} \tag{88}$$

where $N \in \mathbb{N}$.

Additionally the noncommutative sphere has a specific basis $\boldsymbol{P}^n_m$ for $m, n \in \mathbb{Z}$, $n \geq 0$ and $|m| \leq n$, and specific ordering Ω_{S^2}. This was given in detail in [7, 9], although it has a slightly different normalisation there. The elements $\boldsymbol{P}^n_m$ are defined via

$$\boldsymbol{P}^m_n = \alpha^{m-n}\varepsilon^{m-n}\left(\frac{(n+m)!}{(2n)!\,(n-m)!}\right)^{1/2}(\mathrm{ad}_{\boldsymbol{X}_-})^{n-m}(\boldsymbol{X}_+{}^n) \tag{89}$$

where $\mathrm{ad}_{\boldsymbol{u}}(\boldsymbol{v}) = [\boldsymbol{u}, \boldsymbol{v}]$. When written as a formally tracefree symmetric polynomial in $(\boldsymbol{X}_0, \boldsymbol{X}_+, \boldsymbol{X}_-)$, $\boldsymbol{P}^m_n$ is a homogeneous polynomial of order n and is independent of R and ε. This justifies (98), as the spherical harmonics can also be written as formally tracefree symmetric polynomials.

There is a sesquilinear form on $\mathcal{A}_{S^2}$ defined by $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \pi_0(\boldsymbol{u}^\dagger \boldsymbol{v})$ where $\pi_0(\boldsymbol{u})$ is the coefficient of $\boldsymbol{u}$ independent of $\boldsymbol{x}_i$ when $\boldsymbol{u}$ is written as a formally tracefree symmetric polynomial. The sesquilinear form is related to the trace via

$$\pi_0(\boldsymbol{u}) = \mathrm{tr}_{\mathcal{A}}(\boldsymbol{u}) \tag{90}$$

With respect to this sesquilinear form the basis elements $\boldsymbol{P}^n_m$ are orthogonal. Each $\boldsymbol{P}^m_n$ is an eigenvector of the operators $\mathrm{ad}_{\boldsymbol{X}_0}$ and $\Delta = \mathrm{ad}_{\boldsymbol{X}_0}{}^2 + \frac{1}{2}(\mathrm{ad}_{\boldsymbol{X}_+}\mathrm{ad}_{\boldsymbol{X}_-} + \mathrm{ad}_{\boldsymbol{X}_-}\mathrm{ad}_{\boldsymbol{X}_+})$:

$$\mathrm{ad}_{\boldsymbol{X}_0}\boldsymbol{P}^m_n = \varepsilon m \boldsymbol{P}^m_n \tag{91}$$

$$\Delta \boldsymbol{P}^m_n = \varepsilon^2 n(n+1)\boldsymbol{P}^m_n \tag{92}$$

The ladder operators $\mathrm{ad}_{\boldsymbol{X}_+}, \mathrm{ad}_{\boldsymbol{X}_-}$ increase or decrease m:

$$\mathrm{ad}_{\boldsymbol{X}_\pm}\boldsymbol{P}^m_n = \alpha\varepsilon(n \mp m)^{1/2}(n \pm m + 1)^{1/2}\boldsymbol{P}^{m\pm 1}_n \tag{93}$$

and the "normal" of $\boldsymbol{P}^m_n$ is given by

$$\langle \boldsymbol{P}^m_n, \boldsymbol{P}^m_n\rangle = \|\boldsymbol{P}^m_n\|^2 = \alpha^{2n}\frac{(n!)^2}{(2n+1)!}\prod_{r=1}^{n}(4R^2 + \varepsilon^2(1-r^2)) \tag{94}$$

The product of two basis elements is given in terms of Wigner $6j$ symbols:

$$\boldsymbol{P}^{m_1}_{n_1}\boldsymbol{P}^{m_2}_{n_2} = \sum_{n=|n_1-n_2|}^{n=n_1+n_2} C^{n_1\,n_2\,n}_{m_1 m_2 m_1+m_2}\mathcal{R}^{n_1 n_2 n}\boldsymbol{P}^{m_1+m_2}_n \tag{95}$$

where $C^{n_1\,n_2\,n}_{m_1 m_2 m_1+m_2}$ is the Clebsh-Gordon coefficient, and the reduced matrix element $\mathcal{R}^{n_1 n_2 n}$ is given by

$$\mathcal{R}^{n_1 n_2 n} = (-1)^{N+1+n_1+n_2}\frac{\|\boldsymbol{P}^{m_1}_{n_1}\|\|\boldsymbol{P}^{m_2}_{n_2}\|}{\|\boldsymbol{P}^{m_1+m_2}_n\|}(N)^{1/2}(2n_1+1)^{1/2}(2n_2+1)^{1/2}\begin{Bmatrix} \frac{N-1}{2} & n_1 & \frac{N-1}{2} \\ n_2 & \frac{N-1}{2} & n \end{Bmatrix} \tag{96}$$

where $N = (4R^2\varepsilon^{-2}+1)^{1/2}$ and the symbol in the curly brackets is Wigner's 6-j coefficient. Note the right hand side of (96) is only defined when $N \in \mathbb{N}$.

The image of $\varphi_N(\boldsymbol{P}^n_m)$ is a Wigner Operator. This must be written in half integer notation, where $2k+1 = N$ and $j = -k, -k+1, \ldots, k$.

$$\varphi_N(\boldsymbol{P}^m_n)|k,j\rangle = (-1)^n \|\boldsymbol{P}^m_n\|(2n+1)^{1/2}\left\langle \begin{matrix} & n & \\ 2n & & 0 \\ & n+m & \end{matrix}\right\rangle|k,j\rangle \tag{97}$$

As well as the normal ordering and the central ordering there is a "Wick-like" ordering Ω_{S^2} is given by

$$\Omega_{S^2}(\psi^m_n) = (-1)^n \frac{((2n+1)!)^{1/2}}{n!(2R)^n}\boldsymbol{P}^n_m \tag{98}$$

Hence $\boldsymbol{P}^n_m$ may be thought of as the noncommutative analogue of spherical harmonics. We also note that this ordering is compatible with the trace map since $\mathrm{tr}_N(\Omega_{S^2}(u))$ is independent of N. A closed formula for the corresponding star product is being searched. It is known [2] that it is not a differential star product.

3.6 Complex and Other Planes

Lemma 21. *Let $\mathcal{F} = \mathcal{F}^{(\infty)}$ be generated by $\mathfrak{x}, \mathfrak{y}, \varepsilon$. Let $\mathfrak{c}' \in \mathcal{F}$ be any element. Then there is a noncommutative plane given by $\mathcal{A} = \mathcal{F}/\mathcal{I}\,\{[\mathfrak{x},\mathfrak{y}] \sim i\varepsilon\mathfrak{c}'\}$, if $\boldsymbol{c} = \boldsymbol{c}^\dagger$ where $\boldsymbol{c} = Q(\mathfrak{c}')$.*

Proof. Trivial. □

As usual we set $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}$ to be the images of $\mathfrak{x}, \mathfrak{y}$ under the quotient. We note that in general this procedure does not produce an ANCG for $n \geq 2$. This is because in general the Jacobi identity is not satisfied. This is a rich source of ANCGs. If we set $\boldsymbol{z} = \boldsymbol{x} + i\boldsymbol{y}$ and $\overline{\boldsymbol{z}} = \boldsymbol{x} + i\boldsymbol{y}$ then this is often called the noncommutative complex plane. An example is given in [12].

4 An Application: Finite Models of compact surfaces

In this section we give a finite element method for analysing surfaces based on expansions in spherical harmonics. As mentioned in the introduction, this method is based on noncommutative geometry and hence there is an error introduced depending on the order of multiplication. However, the result is associative.

Let us assume that $\mathcal{M}$ is a surface of genus 0 and we have the diffeomorphism $\Psi_\star \colon S^2 \mapsto \mathcal{M}$. From $\Psi_\star$ we generate the pull back map $\Psi^\star \colon C^\omega(\mathcal{M}) \mapsto C^\omega(S^2)$.

To convert the functions $u \colon \mathcal{M} \mapsto \mathbb{C}$ into matrices we would ideally use the homomorphism

$$\Phi_N \colon C^\omega(\mathcal{M}) \mapsto M_N(\mathbb{C})\,; \qquad \Phi_N = \varphi_N \circ \Omega_{S^2} \circ \Psi^\star$$

where Ω_{S^2} is given in (98), and φ_N is given in (88). However, in general, $\Psi^\star(u) \in C^\omega(S^2)$ does not belong to $\mathcal{A}^0_{S^2}$; that is, a finite sum of spherical harmonics. As mentioned in section 2.8 we can still define the image of Φ_N via

$$\Phi_N(u) = \sum_{n,m}\left(\int_{S^2} \overline{\psi^m_n}\Psi^\star(u)\sin\theta d\theta d\phi\right)(-1)^n\frac{((2n+1)!)^{1/2}}{n!(2R)^n}\varphi_N(\boldsymbol{P}^n_m) \tag{99}$$

We can calculate $\varphi_N(\boldsymbol{P}^n_m)$ using (97). Since we have a loss of information when converting from functions to matrices we cannot expect an inverse map. However the "one sided inverse" to Φ_N is given by

$$\Upsilon_N\colon M_N(\mathbb{C}) \mapsto C^\omega(\mathcal{M})\,; \qquad \Upsilon_N(u_N) = \sum_{n,m} \frac{\mathrm{tr}_N(\varphi_N(\boldsymbol{P}^n_m)^\dagger u_N)}{\varphi_N(\|\boldsymbol{P}^n_m\|^2)}(-1)^n \frac{n!(2R)^n}{((2n+1)!)^{1/2}} \Psi^{\star-1}(\psi^n_m) \tag{100}$$

It is easy to show that these satisfy

$$\Phi_N \circ \Upsilon_N = 1_{M_N(\mathbb{C})} \tag{101}$$

$$u - \Upsilon_N(\Phi_N(u)) = \sum_{n=N}^{\infty} \sum_{m=-n}^{n} \left(\int_{S^2} \overline{\psi^m_n} \Psi^\star(u) \sin\theta d\theta d\phi \right) \tag{102}$$

If $\{x_1, x_2, x_3\}$ are the immersion coordinates of $\mathcal{M}$ then $\{\Phi_N(x_1), \Phi_N(x_2), \Phi_N(x_3)\}$ encode the geometry of $\mathcal{M}$ into matrices. Other "external" information on $\mathcal{M}$; that is any function $h\colon \mathcal{M} \mapsto \mathbb{C}$ (for example representing density,) is also encoded as $\Phi_N(h)$.

The next step is to convert the expression for the desired result, in terms of a matrix expression. For this we employ theorem 19 and theorem 11. Theorem 19 states that any differentiation can be written in terms of the Poisson bracket. We can therefore use (19) to give the differentiation in terms of a commutator. Theorem 11 states that we can replace integration with the trace. Combining these we show that if the result can be expressed solely in terms of the known functions via integration and differentiation then we can rewrite the expression as matrix operations. Applying this we obtain the result as a matrix. Finally we apply Υ_N to obtain an approximate result. Clearly the rate of convergence for this algorithm depends on the rate of convergence of the modular expansion of the functions $\{x_1, x_2, x_3\}$ and the external functions h_s.

5 Discussion

We have given a consistent definition of an algebra $\mathcal{A}$ in terms of noncommuting coordinates of an immersion space. When a parameter ε is set to zero, we obtain the commutative algebra $\mathcal{A}^0$ of functions on an algebraic manifold $\mathcal{M}$. This $\mathcal{A}^0$ is a subalgebra of $C^\omega(\mathcal{M})$, which is dense if $\mathcal{M}$ is compact. We have shown that $\mathcal{M}$ inherits a Poisson structure as the limit of the commutator. If we give $\mathcal{A}$ an ordering then we obtain a star product on $\mathcal{M}$. We have define homomorphism and isomorphisms between noncommutative geometries. By mapping one noncommutative geometry to the Heisenberg algebra, we have given an analogue of the coordinate chart and have given $\mathcal{A}$ a quantum group structure. Noncommutative versions of $\mathbb{R}^n$, $T^\star S^2$, T^2, S^2 and surfaces of rotation have been developed. The metric has been extended to noncommutative geometry and used to give an application of noncommutative geometry to the numerical analysis of surfaces.

One of the principle challenges is to enlarge $\mathcal{A}$ so that $\mathcal{A}^0 = C^\omega(\mathcal{M})$. This would enable us to generalise theorem 5 and say that equation (5) is an exact equivalence. We have already suggested how to partially enlarge $\mathcal{A}$ for some examples such as the surfaces of rotation and flat space. One possibility is to use an ordering to define $\mathcal{A}^\star$. If this ordering is chosen so that (1) we can extend the domain of Ω to $C^\omega(\mathcal{M})$ and (2) the star product was a differential star product then we would have such an extension. An alternative, would be an intrinsic definition

of an algebraic noncommutative geometry using the coordinate charts described in section 2.6. This would require a definition of analytic continuation. It would make the definition of a noncommutative manifold independent of the immersion and similar in spirit to the definition of a standard manifold.

We note that if $\mathcal{A}$ has a Banach or $c^\star$ structure, then this could be used to complete $\mathcal{A}$. However in general such a structure does not exists. Thus this approach differs from that of Alan Connes, who investigated an alternative definition of a noncommutative geometry $\mathcal{A}$ so that it was a $c^\star$ algebra. As a result he sets the maps π and Ω so that, [4, page 156]

$$\lim_{\varepsilon\to 0}(\Omega(u)+\lambda\Omega(v)-\Omega(u+\lambda v))=0\,,\qquad \forall u,v\in\mathcal{A}^0\,,\quad \lambda\in\mathbb{C} \tag{103}$$

$$\lim_{\varepsilon\to 0}(\Omega(u)\Omega(v)-\Omega(uv))=0\,,\qquad \forall u,v\in\mathcal{A}^0 \tag{104}$$

$$\lim_{\varepsilon\to 0}(\Omega(u^\dagger)-\Omega(u)^\dagger)=0\,,\qquad \forall u\in\mathcal{A}^0 \tag{105}$$

All these are true for the definition of Ω in this article since $\pi\circ\Omega=1_{\mathcal{A}^0}$. However (103) is true for all ε not just in the limit, and (105) is true for all ε if Ω is a unitary ordering.

As mentioned in the introduction, there are many ways of defining the analogue of a tangent vector field, and we would like to extend the definition of a vector field given in [8, 10] for spheres and surfaces of rotation, to that of a general algebraic noncommutative geometry.

Considering some of the physical applications of this theory; as mentioned, noncommutative geometry has been suggested as a candidate for quantum gravity. Since the classical spacetime inherits a Poisson structure from the noncommutativity of $\mathcal{A}$, we should apply this procedure to spacetimes such as the Schwarzchild black hole where there is a “natural” Poisson structure arising from the Killing-Yano tensors. This will enable one to study the suggestion by ’tHooft and others that the event horizon should contain only a finite quantity of information.

An alternative applications is given in [11], where Gratus and Tucker use an algebra, based on the noncommutative surface of rotation, to describe a Q-brane, a possible model for states of matter. They also suggest how to interpret $\mathcal{M}$ as a phase space even when $\mathcal{M}$ is not a cotangent bundle.

Finally we would like to demonstrate situations in the real world of mathematical modelling, where the method outlined in section 4, is more efficient than standard approaches.

Acknowledgement

The author would like to thank Robin Tucker and Marianne Karlsen for their suggestions and help in the preparation of this article, and the physics department of Lancaster University for their facilities.

References

[1] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Diformation Theory and Quantization I. Deformations of Symplectic Structures. Annals. Phys. **111**, 61-110 (1978)

[2] Cahen, M., Gutt, S.: Non localité d’une deformation symplectique sur la sphere S^2. Bull. Soc. Math. Belg. Ser. B **36** 207-214 (1984)

[3] Connes, A.: Non-Commutative Differential Geometry. Publications of the Inst. des Hautes Etudes Scientifique **62** 257. 1986,

[4] Connes, A.: Noncommutative Geometry. Academic Press 1994

[5] de Wit, B., Marquard, U., Nicolai, H.: Area-Preserving Diffeomorphisms and Supermembrane Lorentz Invariance. Commun. Math. Phys. **128**, 39-62 (1990)

[6] Dubois-Violette, M., Madore, J., Kerner, R.: Shadow of noncommutativity. J. Math. Phys, **39**, 730-738 (1998)

[7] Gratus, J.: A Natural Basis of States for the Noncommutative Sphere and its Moyal bracket. J. Maths. Phys, **38**, 4283 - 4300 (1997) q-alg/9703038

[8] Gratus, J.: A Natural Basis of Vector and Spinor States for the Noncommutative Sphere. J. Maths. Phys, **39**, 2306-2324 (1998) q-alg/9708003

[9] Gratus, J.: Wick Rotations: The Noncommutative Hyperboloids, and other surfaces of rotations. Lett. Maths. Phys, **47**, 97-109 (1999) q-alg/9801036

[10] Gratus, J.: Quantum Topology Change and Vector Modules of Noncommutative Surfaces of Rotations. Submitted to J. Math. Phys. January 1999

[11] Gratus, J., Tucker, R.: A Quantum Geometric Description of a Q-brane with Intrinsic Spin. Submitted to Phys. Rev. Lett. (May 1999)

[12] Klimek, S., Lesniewski, A.: Quantum Riemann Surfaces: I The Unit Disc. Commun. Math. Phys. **146** 103-122, (1992)

[13] Madore, J.: An Introduction to Noncommutative Differential Geometry and its Physical Applications. Cambridge University Press 1995,